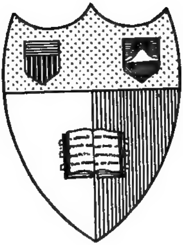
ELEMENTARY MATHEMATICAL ANALYSIS

YOUNG AND MORGAN



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ELEMENTARY MATHEMATICAL ANALYSIS

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ELEMENTARY MATHEMATICAL ANALYSIS

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PREFACE

THIS book aims to present a course suitable for students in the first year of our colleges, universities, and technical schools. It presupposes on the part of the student only the usual minimum entrance requirements in elementary algebra and plane geometry.

The book has been written with the hope of contributing something toward the solution of the problem of increasing the value and significance of our freshman courses. The recent widespread discussion of this problem has led to the general acceptance on the part of many teachers of certain principles governing the selection and arrangement of material and the point of view from which it is to be presented. Among such principles, which have guided us in the preparation of this text, are the following.

1. More emphasis should be placed on insight and understanding of fundamental conceptions and modes of thought, less emphasis on algebraic technique and facility of manipulation. The development of proficiency in algebraic manipulation as such we believe has little general educational value. It is valuable only as a means to an end, not as an end in itself. A certain amount of skill in algebraic reduction is, of course, essential to any effective understanding of mathematical processes, and this minimum of skill the student must secure. But it seems undesirable in the first year to emphasize the formal aspects of mathematics beyond what is necessary for the understanding of mathematical thought. This is especially true for that great majority of students who do not continue their study of mathematics beyond their freshman

year and who study mathematics for general cultural and disciplinary purposes. It seems to us altogether probable, however, that even in the case of students who expect to use mathematics in their later life work (as scientists, engineers, etc.) greater power will be gained in the same length of time, if their first year in college is devoted primarily to the gaining of insight and appreciation, rather than technical facility. Experience has shown only too conclusively that in many cases the emphasis usually placed on formal manipulation effectually prevents the development of any adequate sort of independent power.

2. The reference above to the general cultural and disciplinary aims of mathematical study at once raises the question as to the selection of the material that is to form the content of the course. The cultural motive for the study of mathematics is found in the fact that mathematics has played and continues to play in increasing measure an important rôle in human progress. An educated man or woman should have some conception of what mathematics has done and is doing for mankind and some appreciation of its power and beauty. To this end our introductory courses should cover as broad a range of mathematical concepts and processes as possible. In particular, they must not confine themselves to ancient and medieval mathematics, but must give due consideration to more modern mathematical disciplines. The fundamental conceptions of the calculus must be introduced as early as is feasible in view of the essential rôle they have played in the progress of civilization.

If this broad cultural aim is accepted as one of the fundamental principles in the selection of material, we shall readily agree that much that is now generally considered necessary can and should be eliminated from our general courses in mathematics. Almost all of the conventional course in solid geometry would fall in this category, as well as much of what is now taught as college algebra, all of the more specialized portions of analytic geometry, etc. The time thus gained could then be used for topics that are culturally more significant.

3. The disciplinary motive for the study of mathematics is the one most often emphasized and need not be elaborated here. In spite of much recent criticism of the doctrine of formal discipline in education and in spite of the fact that some of this criticism as applied to mathematics seems to us justified, we firmly believe that faith in the disciplinary value of mathematics is fundamentally sound. Teachers of mathematics need, however, to formulate with precision their aims and purposes in this direction and make their practice conform to this formulation. The disciplinary value of mathematics is to be sought primarily in the domain of thinking, reasoning, reflection, analysis; not in the field of memory, nor of skill in a highly specialized form of activity. We come back here to the conflict between insight and technique discussed earlier in this preface. Suffice it to remark here that the purpose of technical facility is to economize thought, rather than to stimulate it. If our primary purpose is to stimulate thought, we must place the major emphasis on the mathematical formulation of a problem and on the interpretation of the final result, rather than on the formal manipulation which forms the necessary intermediate step; on the derivation of a formula rather than merely on its formal application; on the general significance of a concept rather than on its specialized function in a purely mathematical relation.

If we desire to enhance the general disciplinary value of mathematics, we will seek out and emphasize especially those conceptions and those modes of thought of our subject which are most general in their application to the problems of our everyday life. It is fortunate for our purpose — and it is probably more than a mere coincidence — that the conceptions and processes of mathematics which most readily suggest themselves in this connection are the same that are suggested by the cultural motive discussed earlier. The concept of *functionality* and the mathematical processes developed for the study of functions are precisely the things in mathematics that have most effectively contributed to human progress in more modern times; and the thinking stimulated by this concept and these processes is fundamentally similar to the thought which we are continually applying to our daily problems. "Functional thinking," to use Klein's famous phrase, is universal. It comes into play when we make the simplest purchase, as well as when we attempt to analyze the most complicated interplay of causes and effects.

In the preparation of this text, we have sought to give an introduction to the elementary mathematical functions, the concepts connected therewith, the processes necessary to their study, and their applications. By making the concept of a function fundamental throughout we believe we have gained a measure of unity impossible when the year is split up among several different subjects. The arrangement of this material is exhibited in the table of contents and the text proper, and need not be discussed here. We would merely call attention briefly to some features which seem to require emphasis or explanation.

The change in the value of a function due to a change in the value of a variable is emphasized from the very beginning. The change ratio $\Delta y/\Delta x$ is introduced in Chapter III for the linear function, and the derivative is introduced as the slope of the graph of a quadratic function in Chapter IV, although the word "derivative" is not introduced until Chapter XIX. Derivatives are used in Chapters IV, V, X, XII, XIII, and XIX.

We have discussed rather more fully than is customary those topics which involve new and important concepts, and have been correspondingly brief where we felt the student ought to be able to supply the argument himself. We have tried throughout to place the emphasis on an understanding of the general bearing of the principles, and have consistently tried to minimize difficulties of mere algebraic technique. It seems quite likely that customary classroom procedure will, therefore, need to be modified in the direction of lessening the time given to formal recitations and increasing the opportunities for informal discussion. A number of questions have been inserted among the exercises which it is hoped will stimulate such discussion; this is the purpose also of a number of the "Why's" scattered throughout the text.

The lists of "Miscellaneous Exercises" found at the end of chapters beginning with Chapter XI contain some exercises too difficult for assignment in an average class. These may be used to advantage, we hope, in so-called "honor sections" consisting of men who have shown exceptional ability in mathematics.

A word regarding our conception as to how the text may be applied to meet the varying mathematical preparation of students will not be out of place. At Dartmouth College we propose to distinguish in this connection only two kinds of freshmen: those who enter without trigonometry, and those who have passed a course in trigonometry in their secondary school. The former will cover the first fifteen chapters of this text in a course meeting three hours per week throughout the year (about ninety assignments). These men will have all the necessary preliminary training for the usual courses in the calculus. Those students who enter with trigonometry will cover the first nineteen chapters in a course meeting three hours per week throughout the year, covering the material of Chapters VI, VII, VIII, and IX (which for them is largely review) in about three weeks.

In a course meeting five times per week throughout the year, there should be ample time also for a thorough study of the important topics of Chapter XX (Determinants) and Chapters XXI–XXII (Functions of two independent variables; analytic geometry of space).

So much has been said in recent years in favor of a unified course in mathematics for freshmen that it seems desirable actually to try it out in practice. For this purpose a textbook is necessary. We do not believe that this text will solve the problem; the most we can hope for is that we have secured a first approximation. It is for this reason that we urgently request users of this text to communicate to us any criticisms or suggestions that occur to them looking to the improvement of possible later editions. In particular, we should like advice and counsel as to the possible desirability of increasing the amount of calculus included in the first year. This could be done by devoting less space to the purely geometric aspects of analytic geometry. On theoretical grounds we believe this to be desirable. We felt, however, that we ought to be conservative in case of an innovation of this sort, with a view of seeing how the introduction to this limited extent of the notion of the derivative in the first year fares. If the results are satisfactory, we could then take the next step with confidence.

J. W. YOUNG.
F. M. MORGAN.

HANOVER, N. H.,
April, 1917.

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ELEMENTARY MATHEMATICAL ANALYSIS

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PART I. INTRODUCTORY CONCEPTIONS

CHAPTER I

FUNCTIONS AND THEIR REPRESENTATION

1. The General Idea of a Function. Our daily activities continually furnish us with examples of things that are related to one another, of quantities which depend on certain other quantities, which change when certain other quantities change. Thus, a man's health is related to the food he eats, the exercise he takes, and to many other things. The price of any manufactured article depends on the cost of production, while the latter cost in turn depends on the cost of the raw material, the cost of labor, etc. The weather depends on a variety of conditions. These are complicated examples of dependence. There are very simple examples. Thus the price paid for a certain quantity of sugar depends on the number of pounds bought and the price per pound; the area of a square depends on the length of one of its sides; and so forth.

In all such cases, where some quantity depends on some other quantity or quantities, we say that the former is a ***function*** of the latter. Thus the price of an article is a function of the cost of production, the area of a square is a function of the length of one of its sides, etc.

2. General Laws. Many problems of science consist in expressing as accurately as possible one quantity in terms of another quantity on which the first depends. The statements, "The area of a square is equal to the square of the length of one side," and "The speed of a body falling from rest is proportional to the time it has fallen" are simple examples.

At the basis of this idea of dependence or functionality is the notion of a ***general law*** which the quantities in question obey. Most of the problems of civilized life are concerned, directly or indirectly, with the investigation of such laws. Thus medical science seeks to discover the laws governing health, economics investigates the laws governing the production and distribution of wealth, the business man studies the conditions which influence his business and his profits. In every case the investigation of the law in question involves finding out how something is related to, depends on, changes with, something else; *i.e.* the study of a function of some kind.

The ability to think clearly about such relationships is of the highest importance to every one. This course in mathematics is primarily concerned with the study of certain of the simpler kinds of functions and their applications.

3. Numbers and Quantities. We shall confine ourselves in general to the study of relations between things which can be *measured.* We can then always speak of the *amount* of one of them. Such an amount is expressed, in terms of a suitable *unit of measure,* by means of a number. Anything that can be represented by means of a number we shall call a ***quantity.***

A function expressing the relation of one such quantity to another gives rise to a relation between numbers. A very powerful aid in studying functions is their *geometric representation,* which we shall discuss presently. We must consider first, however, the geometric representation of a single quantity.

4. The Arithmetic Scale. The distinction between two of the simplest kinds of quantities can be illustrated by reference to their *geometric* or *graphic representation.* Every one is familiar with the so-called ***arithmetic scale*** (Fig. 1), of which the

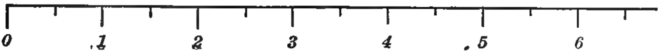


FIG. 1

yard stick and tape measure are examples. The divisions of the scale in these cases represent lengths. Another example is the beam on a certain kind of balance; here the divisions of the scale represent weights.

A characteristic feature of an arithmetic scale is that it begins at some point *O* and extends from *O* in *one* direction. The quantities represented by such a scale are expressed by means of the numbers of arithmetic. These in turn represent simply the *magnitude,* or the *size,* or the *amount,* of something (as 12 yd. of cloth, 96 lb. of sugar, etc.).

5. The Algebraic Scale. Hardly less familiar nowadays is the so-called ***algebraic scale*** (Fig. 2). The most familiar example is probably the scale on an ordinary thermometer. Every one knows the meaning of + 10° or − 5°.

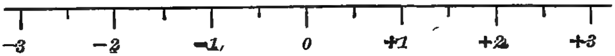


FIG. 2

Such an algebraic scale extends in *two opposite directions* from some *arbitrary* point (marked 0) of the scale. The quantities represented by the points of such a scale are expressed by means of the so-called ***real numbers of algebra,*** such as:

$$\cdots, -4, -\sqrt{12}, -3, -\tfrac{1}{2}, 0, +1, +\tfrac{49}{16}, \cdots.$$

Such a number represents not merely a magnitude, but rather *a magnitude and one of two opposite directions* or *senses.* These two opposite "senses" are of various kinds according to the quantities considered. They are often expressed by such phrases as: "to the right of" and "to the left of," "above" and "below," "greater than" and "less than," "before" and "after," etc. Thus + 10° of temperature means a temperature 10° *greater than* the arbitrary temperature which we have agreed to indicate by 0°; whereas – 5° means a temperature 5° *less than* the temperature indicated by 0°. It should be noted that 0° of temperature does not mean the absence of temperature.

6. Magnitudes and Directed Quantities. We have seen in the last two sections that a number may represent simply a *magnitude;* or, that a number may represent *a magnitude and one of two opposite directions.* The numbers of arithmetic serve the former purpose, the positive and negative numbers of algebra serve the latter. Thus the number 5 represents simply a magnitude, such as a distance of five miles between two stations or a period of time of five hours. The numbers + 5 and – 5 also represent magnitudes of five units; but they represent more than this. They may tell us, for example, that a station is five miles *east of* a certain place denoted by O and that another station is five miles *west of* the place denoted by O, respectively; or that an event took place five hours *after* or five hours *before* a certain event.

We may then distinguish two kinds of quantities: (1) *magnitudes,* and (2) so-called *directed quantities.* Examples of the former are: the length of a board, the weight of a barrel of flour, the duration of a period of time, the price of a loaf of bread, etc. Examples of the latter are: the temperature (a certain number of degrees above or below zero), the distance and direction of some point A on a line from some other point B on the line, the time at which a certain event occurred (a certain number of hours before or after a given instant); etc.[*]

Geometrically, the distinction between directed quantities and mere magnitudes corresponds to the fact that, on the one hand, we may think of the line segment AB as drawn from A to B or from B to A, and, on the other hand, we may choose to consider only the length of such a segment, irrespective of its direction. Figure 3 exhibits the geometric representation of 5, $+5$, and -5. A segment whose direction is definitely taken account of is called a ***directed segment.*** The magnitude of a directed quantity is called its ***absolute value.*** Thus the absolute value of -5 (and also of $+5$) is 5.

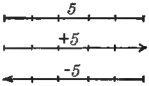


FIG. 3

7. Further Remarks concerning Scales.

Scales, both arithmetic and algebraic, occur in practice in a variety of forms. We have hitherto considered only the simplest form, in which the scale is constructed on a *straight line* and in which the subdivisions corresponding to the numbers 1, 2, 3, $\cdots$ (and in case of the algebraic scale also those corresponding to the numbers $-1, -2, -3, \cdots$) are at *equal intervals.* Neither of these two conditions is essential. A scale may be constructed on a curved line (a circle, for example, in which case it is sometimes called a *dial*). Scales are also used in which the intervals between the points representing the whole numbers are not equal. Such a scale is called a *non-uniform* scale. The scales on some forms of thermometers, on a slide rule (see p. 252), on certain types of ammeters and pressure gauges, etc., may serve as examples of non-uniform scales. The scales discussed in §§ 4, 5 are then to be described more fully as *rectilinear* and *uniform.* In the future, unless specifically stated otherwise, a scale will always mean a uniform scale.

---

[*] We are here considering only magnitudes in one of two opposite directions It is also possible to consider as quantities magnitudes taken in any direction in a plane or in space. Thus a force has a certain magnitude and is exerted in a certain direction; it could be completely represented by a line segment whose length represents the magnitude of the force and whose direction (shown by arrow-head) represents the direction in which it acts. Such quantities are called ***vectors.*** We shall have occasion to refer to them again (Chap. XVIII).

8. Use of Line Segments to Represent Quantities. Statistical Data. A common use of line segments to represent quantities is in connection with the graphic representation of statistical data. The table below, for example, gives the areas of the New England States; the adjacent figure represents these areas by means of line segments.

AREA OF NEW ENGLAND STATES

States	Maine	Vermont	New Hampshire	Massachusetts	Connecticut	Rhode Island
Square Miles	33,040	9,565	9,305	8,315	4,990	1,250

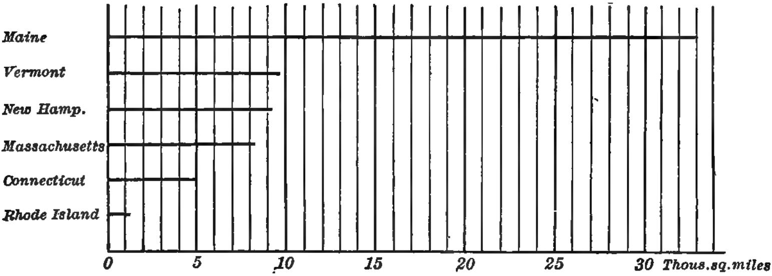


FIG. 4

The method of constructing such a graphic representation should be clear without further comment.

The above areas could also be represented by areas, as in the following figure.

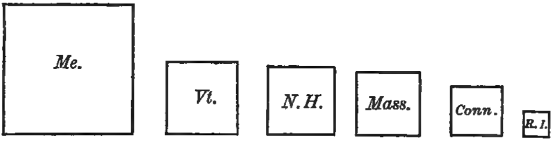


FIG. 5

In general, this method of representation is not so serviceable. Why?

EXERCISES

1. From the following table represent graphically by means of line segments the enrolment in Dartmouth College during the years 1901–1916:

'01–'02	'02–'03	'03–'04	'04–'05	'05–'06	'06–'07	'07–'08	'08–'09
686	709	802	857	927	1058	1131	1136

'09–'10	'10–'11	'11–'12	'12–'13	'13–'14	'14–'15	'15–'16
1197	1165	1242	1246	1284	1336	1422

Use a convenient unit to represent 100 students (say $\frac{1}{4}$ in.). Can you then represent the data with complete accuracy? Why?

2. Represent graphically the size of the libraries of the following institutions:

	No. of Volumes		No. of Volumes
Harvard	1,180,000	Williams	80,000
Yale	1,000,000	Amherst	110,000
Dartmouth	130,000	Wesleyan	100,000
Brown	115,000	Univ. of Vermont . .	91,000

3. Take the edge of a sheet of paper and mark on it a point A. Place this edge along the segment representing the area of Vt. in Fig. 4, the point A coinciding with the left-hand extremity of the segment. Mark the right-hand extremity by a point B on the paper. Do the same with the segment representing N. H., placing the point B at the left-hand extremity, however, and obtaining a new point C, corresponding to the right-hand extremity. Continue this process for the states Mass., Conn., and R. I. The total segment represents the sum of the areas. Show that Me. has an area almost as great as that of the other N. E. states combined. The process just described in the above exercise is known as *graphic addition*.

4. Describe a similar process for graphic subtraction.

5. Show that the distance between two points of an arithmetic scale can always be found by subtraction. Is the same true for the points of an algebraic scale? What is the meaning of the sign of the difference?

6. Two algebraic scales intersect at right angles, the point of intersection being the point 0 of each scale, and the units on both scales being the same. Show how to find the distance from any point on one scale to any point on the other. Would your method still be applicable, if the units on the two scales were different? Explain your answer.

7. In constructing Fig. 5 what theorem of plane geometry regarding the areas of similar figures is used? Could the result of Ex. 3 have been readily obtained from the representation in Fig. 5?

9. The Investigation of Functions. We are now ready to consider in some detail a few special examples of functions, in order to familiarize ourselves with certain general characteristics a function may possess, with certain methods for the representation and study of functions, and with the terminology. This is desirable before taking up the more systematic study of general types of functions.

10. Example I. *The temperature as a function of the time.* The temperature at a given place is a function of the time of day. At any given time we can determine the temperature by

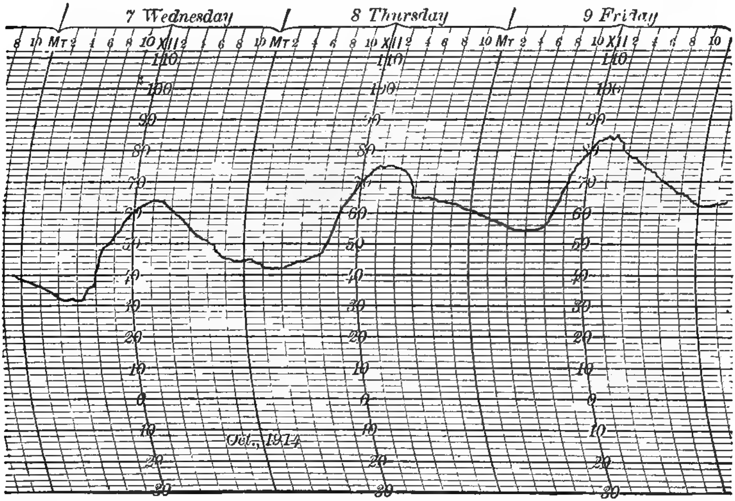


FIG. 6

simply reading an ordinary thermometer. For the meteorologist, however, the actual temperature at any instant is of less importance than the *changes* in the temperature that take place during a period of time (such as a day, a month, etc.). To trace these changes he must know the temperature at every instant. For this purpose he makes use of a self-recording thermometer. A portion of a record of such a thermometer is given in Fig. 6.

The way in which such an instrument works is briefly as follows. The pivoted lever shown in the figure (Fig. 7) carries a pencil point. The mechanism of the instrument causes the pencil end of the lever to rise or fall as the temperature rises or falls, so that if a vertical thermometer scale* were adjusted behind the pencil point we could read off the temperature on this scale. The pencil point rests against a strip of paper, ruled as in Fig. 6, which is mounted on a drum. Clockwork causes this drum to rotate uniformly at the proper speed. The rulings on the strip of paper now explain themselves. The distance between two successive horizontal lines corresponds to 2° of temperature. The distance between two successive vertical arcs corresponds to two hours. The temperature at any instant can then be read from the record on the strip of paper.

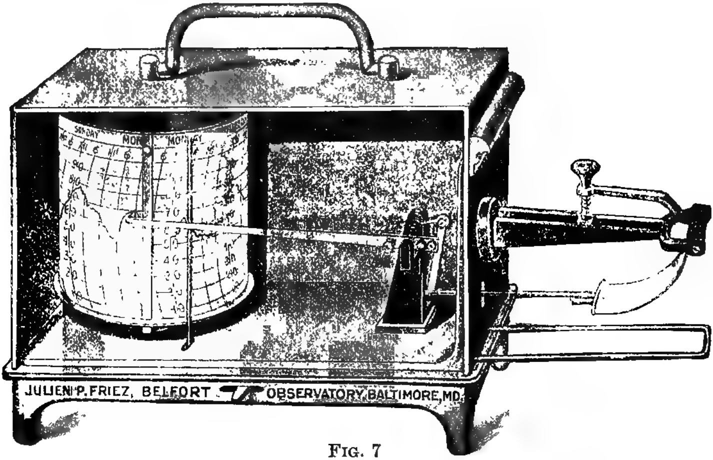


FIG. 7

The way in which such a record may be used is illustrated by the following questions, which refer to the record of Fig. 6.

---

* Since the pencil moves on an arc of a circle, this vertical scale is conveniently constructed on such an arc, rather than on a straight line.

1. What was the temperature at noon on each of the three days given?

2. What was the temperature at midnight between Wednesday and Thursday? At 6 P.M. on Friday?

3. What was the maximum and the minimum temperature on each of the three days, and at what times did it occur?

4. When was the temperature 50°? During what periods was it above 50°?

5. How would a stationary temperature be recorded? A rapidly rising temperature? A rapidly falling temperature?

6. By how many degrees did the temperature change on Wednesday from noon to 2 P.M.? Was this change a rise or a fall?

7. During what two hours on these three days did the greatest rise in temperature occur?

8. When did the most rapid rise in temperature occur? When the most rapid fall?

9. What was the average rate of increase (in degrees per hour) in the temperature from the minimum on Thursday to the maximum on Thursday? The average rate of decrease from the maximum on Wednesday to the following minimum?

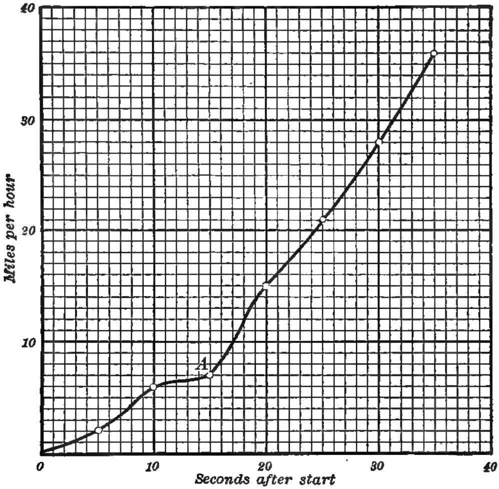
11. Graphic Representation. In the preceding example we exhibited the temperature as a function of the time by means of a curve drawn with reference to a time scale and a temperature scale. Such a curve is called a *graph* of the function in question. Such a *graphic representation* gives a vivid picture of the function; but it is limited in accuracy. Why? Can a change in temperature of 0.1° be distinguished on this graph?

12. Example 2. *Speed in terms of the time.* Readings of the speedometer of an automobile taken every five seconds from a standing start are given in the following table:

Number of seconds after start	5	10	15	20	25	30	35
Speed in miles per hour	2	6	7	15	21	28	36

We proceed to construct a graph of the function thus obtained, as follows. We take a piece of square-ruled paper and on one of the horizontal lines (which for convenience we draw more heavily) construct a uniform scale to represent the time

(Fig. 8). On the vertical lines through the points representing 5, 10, 15, 20, . . . seconds we lay off segments to represent the speeds at the respective instants. This is most conveniently done by constructing on the vertical line through O a scale representing speed in miles per hour. Thus, by reference to the scale indicated in the figure, the point A represents the corresponding values: 15 seconds and 7 miles per hour. The other points indicated in the figure are now readily located, or "plotted," in similar fashion. The final step in constructing the figure consists in drawing a "smooth curve" through the points.



SPEED OF AN AUTOMOBILE

FIG. 8

The curve thus obtained may be used as was the temperature curve discussed in the previous example. We might, for example, conclude from this figure that the speed of the car at the end of 23 seconds was *probably* about $18\frac{1}{3}$ miles per hour.

The necessity of saying "probably", however, exhibits an essential difference between this example and the former one. In case of the temperature record the temperature at every instant was automatically recorded; any point of the curve in that example was as significant as any other point. In the present example the only speeds actually measured are those specifically listed in the above table. And yet the conclusion stated above regarding the speed of the car at the end of 23 seconds is justified. Why?

1. What was the probable speed of the car at the end of 27 seconds?
2. How long did it take the car to pick up from 0 to 30 miles per hour?
3. The driver probably shifted gears between the 10th and 15th seconds. What can be said of the reliability of the curve during this interval?
4. How is the steepness of the curve related to the rate at which the speed is increasing?
5. Is it possible to calculate, by the use of this figure, approximately how far the car traveled during the first 35 seconds?

13. Variables. It is desirable to introduce at this point a certain terminology. In the preceding examples we have considered temperature and speed as functions of (*i.e.* dependent on) the time. We have considered several different instants of time and the corresponding values of the temperature and the speed. Whenever, in a given discussion, we consider a number of different values of a quantity, such as time, or temperature, or distance, or weight, etc., we call such a quantity a ***variable.*** In the above examples, the time and the temperature and the speed are all variables; and, since in the first example we have thought of the temperature as depending on the time, we may speak of the temperature as the *dependent variable*, of the time as the *independent variable.* It is often more convenient, however, to call the dependent variable simply *the function* and the independent variable *the variable*. Thus, in the second example, the speed was the function and the time was the variable.

14. Tabular Representation. Interpolation. In the second example we secured data concerning a function by measurement and exhibited the corresponding values of variable and function by means of a table of values. Such a table is called a *tabular representation* of the function. The accuracy of such a representation is limited only by the precision of measurement. Such a table, however, gives an incomplete description of the function. Why? The process of obtaining values of the function for values of the variable that lie between the recorded values stated in the table is called ***interpolation.*** When the interpolated values are read from a graph of the function, the process is known as *graphic interpolation.* The answers to the first two questions at the end of § 12 were obtained by graphic interpolation.

15. Example 3. *Volume of water as a function of the temperature.* When 1000 cc. of water at 0° centigrade is heated, it is found that the volume of the water changes according to the following table.

Degrees Centigrade	0	2	4	6	8
Cubic Centimeters	1000.00	999.90	999.87	999.90	999.98
Degrees Centigrade	10	12	14	16	20
Cubic Centimeters	1000.12	1000.32	1000.57	1000.86	1001.61

It requires a rather careful examination of this table to learn that as the temperature (the variable) is increased from 0° the volume of the water (the function) decreases and then increases. A graphic representation of this function, analogous to the examples already considered in §§ 10, 12, would have yielded this result at a glance. It is our next concern to see how such a representation can be constructed, in this case.

To this end we a take a piece of square-ruled paper and on one of the horizontal lines construct a uniform scale to represent temperatures. At the points representing 0°, 2°, 4°, 6°, ···, we would then lay off on the vertical lines distances that are to represent the volumes in which we are interested. However, at this point a difficulty presents itself. The numbers representing the volumes in question are so large, and the differences between the volumes for the various temperatures so small, that, if we choose the unit on the vertical scale small enough to represent these volumes on a sheet of paper of convenient size, it would be a practical impossibility

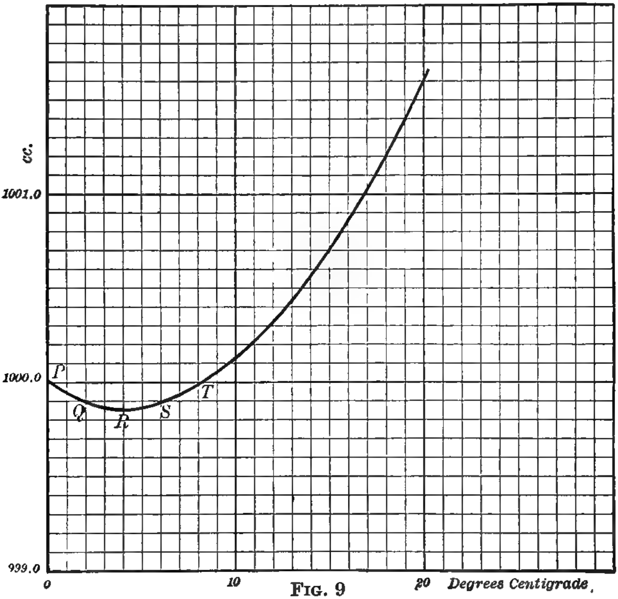


FIG. 9

to represent the volumes with sufficient accuracy to make the differences in the volumes distinguishable. It is precisely these *variations* in volume, however, in which we are primarily interested.

To overcome this difficulty, we adopt the expedient of exhibiting merely that portion of the graphic representation in which we are primarily interested, and are then able to use a largely magnified scale. That is, we observe that all the volumes in which we are interested lie between 999.00 cc. and 1001.00 cc. We may then assume that the points on the line on which we constructed the temperature scale are at a height representing 999.00 cc. (Fig. 9). In other words we suppose the zero point of the vertical volume scale to be a great distance below the point at which we are working. We construct a portion of the volume scale on the vertical line through *O*, marking the latter point 999.0 and choosing the unit on this scale sufficiently large to meet our requirements. In the figure, as drawn, each of the vertical divisions represents 0.1 cc. The construction of the points *P*, *Q*, *R*, ⋯ is then readily made. A smooth curve drawn through the points thus plotted then gives the graph of the function.

Here, again, the points in the curve between the points given by the table are uncertain; but the regularity with which the given points are arranged together with the nature of the phenomenon we are considering leaves little room for doubt that, if the volumes for 1°, 3°, 5°, ⋯ should be measured and the resulting volumes plotted, the resulting points would be located upon (or at least very near to) the curve drawn.

1. What is the volume of water at 7°? at 19°?

2. What is the minimum volume, and at what temperature does it occur?

3. At what temperature besides 0° is the volume 1000.00 cc.?

EXERCISES

1. The following temperatures were observed at Hanover, N.H., on a certain day in February, 1914:

Midnight	− 12° F.				
1 A.M.	− 13°	9 A.M.	− 12° F.	5 P.M.	+ 18° F.
2 A.M.	− 14°	10 A.M.	− 2°	6 P.M.	+ 11°
3 A.M.	− 15°	11 A.M.	+ 4°	7 P.M.	+ 6°
4 A.M.	− 17°	Noon	+ 10°	8 P.M.	+ 2°
5 A.M.	− 20°	1 P.M.	+ 12°	9 P.M.	+ 1°
6 A.M.	− 21°	2 P.M.	+ 14°	10 P.M.	0°
7 A.M.	− 22°	3 P.M.	+ 19°	11 P.M.	− 2°
8 A.M.	− 19°	4 P.M.	+ 22°	Midnight	− 4°

Plot the corresponding points on square-ruled paper, and draw an approximate graph of the function. Assuming this graph to be correct, what was the temperature at 6.30 A.M.? At 6.30 P.M.? What was the total range (the difference between the maximum and the minimum) of temperature during the day? How long did it take the temperature to rise from its minimum to its maximum? At what average *rate* in degrees per hour did the temperature rise during this period?

2. A stiff wire spring under tension is found experimentally to stretch an amount d under a tension T as follows:

T in lb.	10	15	20	25	30
d in thousandths of in. .	8	12	16.3	20	23.5

Plot the above data. What would the stretch be when the tension is 12 lb.? 27 lb.? 23 lb.?

3. The intercollegiate track records are as follows, where d is the distance run and t is the time:

d	100 yd.	220 yd.	440 yd.	880 yd.	1 mile	2 miles
t	$9\frac{4}{5}$ sec.	$21\frac{1}{5}$ sec.	48 sec.	1 m. $53\frac{2}{5}$ sec.	4 m. $14\frac{2}{5}$ sec.	9 m. $23\frac{4}{5}$ sec.

Plot these records by points in a plane, and draw a smooth curve through them. Are the points of this curve significant? Why? What would you expect the record for 600 yd. to be? For 1500 yd.? For 1000 yd.? Compare the results of these interpolations with the actual records for these distances.

4. The following table shows the distance at which objects at sea-level are visible from certain elevations:

ELEVATION FEET	DISTANCE MILES	ELEVATION FEET	DISTANCE MILES	ELEVATION FEET	DISTANCE MILES
1	1.3	40	8.4	200	18.7
5	3.0	50	9.3	300	22.9
10	4.2	100	13.2	500	29.6
20	5.9	150	16.2	1000	33.4
30	7.2				

Plot the graph of this function. Use a different scale for elevation for values from 100 to 1000 ft. from that used from 1 to 50 ft. Why?

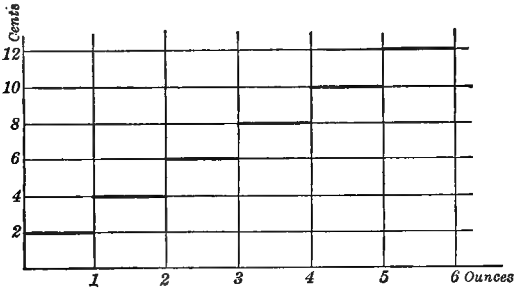
5. The following is an extract of the mortality table prescribed by statute in most states as the basis on which the reserves of life insurance companies shall be computed:

AGE	NUMBER LIVING	AGE	NUMBER LIVING	AGE	NUMBER LIVING
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,237
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

Draw the mortality curve. Of 100,000 living at the age of 10 years approximately how many would be alive at 32 years? At 57 years? How would you represent on the graph the number dying during any given period of five years?

16. Empirical Functions and Arbitrary Functions. The examples of functions we have hitherto considered have been taken from observed measurements of relations existing in nature and life about us. Such functions are called *empirical.* Another type of functions may now engage our attention. They may be called *arbitrary* or *artificial.* The following will serve as an example.

17. Example 4. *Letter postage.* According to the postal regulations the postage on letters is fixed at two cents per ounce or fraction thereof. The graph showing the relation between the amount of postage and the weight of the letter is then given by Figure 10.



Letter Postage

Fig. 10

18. Constant Functions. Continuous and Discontinuous Functions. The graph just referred to exhibits two peculiarities that we have not yet had occasion to observe in connection with a function.

(1) The value of the function may make a sudden jump as the variable passes through certain values (in this case when the weight passes through the values 1 oz., 2 oz., etc.) without taking on the intermediate values. In the present case, as the weight is increased from exactly 1 oz. to the slightest amount

above 1 oz. the postage jumps from 2 cents to 4 cents. A function with such breaks, or changes of a definite amount for no matter how slight a change in the variable, is said to be *discontinuous* for those values of the variable at which the break or jump occurs.

A function, on the other hand, whose graph is a continuous line or curve without such sudden breaks or changes is said to be a *continuous* function.*

(2) Portions of this graph are *horizontal straight lines*, which means that certain changes in the variable produce no corresponding change in the value of the function. Thus, the postage does not change as the weight of the letter is increased from slightly more than 1 oz. to 2 oz. In such a case we say that the function is *constant* (or stationary) for the interval of the variable in question.

We should observe, further, that the graph of the function as drawn does not furnish a unique value for the function at the points of discontinuity, *i.e.* when the weight is 1, 2, 3, ··· oz., since there is nothing to indicate whether we should take the lower or the higher value. As a matter of fact the *arbitrary definition* of the function specifies that the lower value is to be taken.

19. More about Arbitrary Functions. We must not assume, of course, from the preceding example that every arbitrary function is discontinuous.

In fact, we should note that if we take any square-ruled paper, construct on it a horizontal scale, any number of which we will designate by x, and a vertical scale, any number of which we will call y, and then draw an arbitrary curve across the paper, as in Fig. 11, we thereby define a relation between the numbers x of the horizontal scale and the numbers y of the vertical scale, such that to every value of x corresponds a certain value (or possibly a set of values) of y; *i.e.* we define y as a function of x.* The reason for the phrase in parentheses in the last sentence is as follows. If the curve we draw is such that for any value of x the corresponding vertical line cuts the curve in more than one point, there will be associated with such a value of x more than one value of y (Fig. 12). The variable y is in such a case still a function of x, since the values of y are determined by the values of x. The distinction between functions of the latter type and those previously considered is made by the following definitions:

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* The word *continuous* is used in mathematics in a highly technical sense, the full discussion of which is beyond the scope of an elementary course. The definition of the term given above is sufficiently precise for our present purposes. Later we shall have more to say of it.

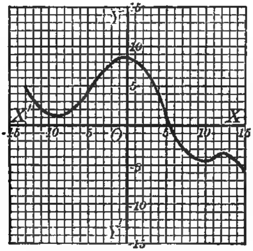

Fig. 11

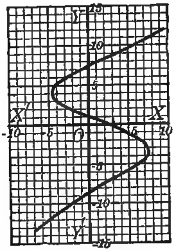

Fig. 12

If to every value of the variable under consideration there corresponds a single value of the function, the function is said to be *single-valued* or *one-valued.* If to any value of the variable corresponds more than one value of the function, the latter is said to be *multiple-valued.*

---

* The *accuracy* with which a function is defined by its graph depends on the accuracy with which it is possible to read the two scales of reference and the "fineness" of the curve.

We shall for the present be concerned primarily with one-valued functions only, although one example of a two-valued function will occur soon. Multiple-valued functions will be considered later (Chapter X).

EXERCISES

1. From the following data construct a graph showing the cost of domestic money orders in the United States:

Amount of Order	Rate	Amount of Order	Rate
Not over $2.50	3 cents	Over $30.00 to $ 40.00	15 cents
Over $ 2.50 to $5.00	5 cents	Over 40.00 to 50 00	18 cents
Over 5.00 to 10.00	8 cents	Over 50.00 to 60.00	20 cents
Over 10.00 to 20.00	10 cents	Over 60.00 to 75.00	25 cents
Over 20.00 to 30.00	12 cents	Over 75.00 to 100.00	30 cents

2. Draw a figure showing the rates for parcel-post packages for zone 1; for zone 2; for zone 3. Compare these graphs.

3. Draw a figure to represent the cost of gas in your own city. Is there a different rate for large consumers? If so, will this show clearly on the graph? How?

4. On a piece of square-ruled paper draw graphs of continuous functions which are rapidly increasing; rapidly decreasing; slowly increasing; slowly decreasing.

5. Draw the graph of an arbitrary function which is increasing and in which the rate at which it increases is increasing. Also that of an increasing function in which the rate of increase is decreasing.

20. Analytic Representation of Functions. We have hitherto considered two methods of representing a function, the graphic and the tabular. There is a third method, called the *analytic*, which in its simplest form consists of the expression of the function in terms of the variable by means of a *formula*, from which the corresponding values of the variable and the function can be *computed*. The following will serve as examples.

21. Example 5. *Capital and interest.* The *amount* A in t years of $ 1000 drawing simple interest of 5 % is given by the formula

(1) $A = 1000 + 50t.$

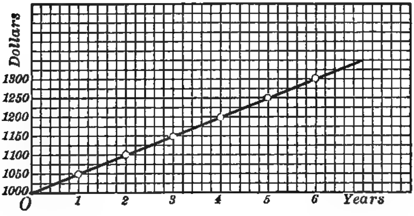

FIG. 13

By substituting for t a succession of values and computing the corresponding values of A, we obtain from this formula a tabular representation of the function. This in turn can be represented graphically. The table above and Fig. 13 are the result.* The points plotted appear to be on a straight line. Prove that they are.

t (years)	0	1	2	3	4	5	6
A (dollars) . . .	1000	1050	1100	1150	1200	1250	1300

22. Example 6. *The area of a square.* The area (in square inches) of a square whose side is x inches long is given by the formula

$$y = x^2.$$

From this equation, we readily compute the following table.

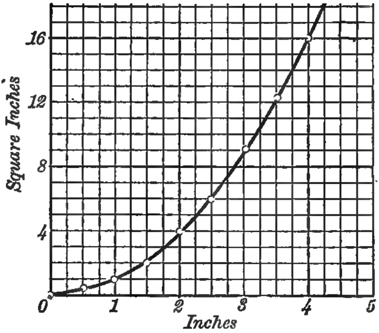

FIG. 14

x (in.) . .	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y (sq. in.) .	0	0.25	1.00	2.25	4.00	6.25	9.00	12.25	16.00

---

* In practice bankers do not take account of fractions of a day in computing interest. Strictly speaking, therefore, the graph of the function A, *as used in practice*, is discontinuous. This practice of bankers is, however, dictated by convenience. It does not alter the fact that the function, as such, is continuous.

Using these values, it is now easy to draw the graph, which is shown in Fig. 14.[*]

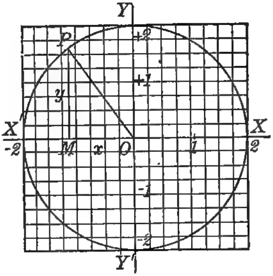

FIG. 15

23. Example 7. *The function defined by a circle.* It is often desirable to obtain an analytic representation of a function, originally given graphically or by means of a table. Such an analytic representation is sometimes easy to obtain. Suppose, for example, that on square-ruled paper an x-scale and a y-scale have been constructed *with the units on the two scales equal*, and suppose that with the common 0-point of the scales as a center a circle is drawn with a radius of 2 units (Fig. 15). The functional relation between the variables x and y defined by this curve is to be expressed by means of a formula.

If P is any point on the circle, the absolute values of the x and the y of this point form the legs of a right-angled triangle of which the hypotenuse measures 2 units. By a well-known theorem of geometry we have then $y^2 = 4 - x^2$ or

$$y = \pm \sqrt{4 - x^2}.$$

This is the analytic representation sought. It may be noted that we have here to do with a two-valued function.

24. Range of a Variable. We had occasion some time ago (§ 13) to introduce the term *variable*. In the future such a quantity will generally be represented by a symbol, such as x, or y, or t, etc. Indeed this was done in some of the preceding examples. The various values attached to such a symbol throughout the discussion are numbers. These numbers constitute the *range* of the variable in question.

The range of a variable is usually determined by the nature of the problem under consideration. Often it is very definitely *restricted*. Thus in the case discussed in the last article the range of x (as well as that of y) consists of all (real) numbers from -2 to $+2$, and no others. For numbers outside this range, the function in question is not defined. Again, in the case of the mortality table considered in Ex. 5, p. 17, the range of the dependent variable (the number of persons living at a given age) is restricted to whole numbers less than 100,000; fractional values of the variable are here meaningless.

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[*] When, as here, the only fractional parts of a unit which occur are halves, quarters, etc., it is convenient to use a ruled paper on which the larger units are subdivided into *four* or *eight* parts instead of *ten*.

25. Increasing and Decreasing Functions. A function which increases when the variable increases is called an *increasing* function; if, on the other hand, the function decreases as the variable increases, the function is called *decreasing.* Thus the amount A of capital and interest recently considered is an increasing function of the time t, throughout the range of the latter. Also, the area of a square is an increasing function of the length of one of its sides. On the other hand, the number of people living at a given age is a decreasing function of the age. A function may be increasing for certain values of the variable and decreasing for certain other values. Thus, the temperature is during certain portions of the day an increasing function, during other portions a decreasing function. The volume considered in § 15 is a decreasing function of the temperature T, from $T = 0$ to $T = 4$, and an increasing function for values of T greater than 4.*

If the two scales with reference to which the graph of a function is constructed are placed in the more usual way, so that the numbers on the scales increase to the right and upward, respectively, what distinguishes the graph of an increasing function from that of a decreasing one?

---

[*] In the case of the circle discussed in § 23, the function has two "branches" in the interval from $x = -2$ to $x = +2$, the one consisting of the positive values of y, the other of the negative values of y. The function may be considered as consisting of two one-valued functions, one of which increases from $x = -2$ to $x = 0$ and decreases from $x = 0$ to $x = +2$, while the other decreases from $x = -2$ to $x = 0$ and increases from $x = 0$ to $x = +2$.

EXERCISES

1. If a body falls from rest, its speed v in feet per second at the end of t seconds is given by the relation $v = 32\,t$. Construct the graph of v as a function of t.

2. The charge for printing n hundred circulars of a certain kind is $p = 2\,n + 10$ dollars. Represent the function graphically.

3. The express rate r on a package is computed from the following formula: $r = \frac{w}{100}(p - 30) + 30$, where w is the weight of the package in pounds and p is the charge per hundred pounds. Draw the graph of r as a function of w, for each of the values $p = 40, 50, 80, 100$. What comment would you make on this rule for $p = 30$, or for values of p less than 30? This is an example in which the range of the variable is arbitrarily limited to be not less than a certain amount. The formula in this exercise really gives r as a function of the *two variables* w and p.

4. When a body is dropped from a height of 200 ft., its distance s from the ground at the end of t sec. is given by $s = 200 - 16.1\,t^2$. Draw the graph of s as a function of t. In how many seconds will the body reach the ground? At what time is the speed of the body greatest? Least? What relation has the steepness of the graph to the speed of the body? Why? What are the natural limitations on the range of the variable?

5. In Fig. 13, the beginning of the A-scale does not appear on the graph. Why?

6. ***Rate of increase.*** In the function of § 21, when $t = 2$, we have $A = 1100$. Starting with this *initial value* of t, let t be increased by 1, by 2, by 3, ⋯ The corresponding values of A (*i.e.* the values of A when $t = 2 + 1 = 3$, $2 + 2 = 4$, etc.) are respectively 1150, 1200, 1250, ⋯, and the corresponding *increases* in A over the initial value 1100, are 50, 100, 150, ⋯. We see then that for these values the increase in A is always equal to 50 times the corresponding increase in t.* Show that the same is true if we start with a different initial value of t, say $t = 3$. Prove, *in general*, that starting with *any* particular value, say $t = t_1$, of t, and *any* increase in t, say an increase equal to h, that the resulting increase in A is equal to $50\,h$; *i.e.* that the ratio

$$\frac{\text{increase in } A}{\text{corresponding increase in } t} = 50.$$

---

* When a change in the value of the variable produces a certain change in the value of the function, these two changes correspond *to each other*. We may then speak of either change as corresponding to the other.

7. From the result of Ex. 6, show that the graph of the function there considered is a straight line.

8. Make an investigation similar to that in Ex. 6 for the function $y=x^2$ considered in § 22; *i.e.* calculate the increase in y due to an increase in x, under a variety of conditions. For example, let $x = 2$ initially, and calculate the increases in y resulting from increases of 0.5, 1.0, 1.5, 2.0 in x. For each case calculate the ratio:

$$\frac{\text{increase in } y}{\text{corresponding increase in } x}.$$

Is this ratio constant? Is the increase in y due to an increase in x of 1.0 the same when the initial value of x is 3 as it is when the initial value of x is 2? How is the change in the steepness of the graph related to your result?

9. A car begins to move and gradually increases its speed in such a way that in x sec. it has traveled $y = x^2$ ft. Interpret in this new setting the "increase in y due to a certain increase in x," as computed in the preceding exercise. Show in particular that the "increase in y" is the distance traveled by the car during the interval of time represented by the corresponding "increase in x," and that the ratio

$$\frac{\text{increase in } y.}{\text{corresponding increase in } x}$$

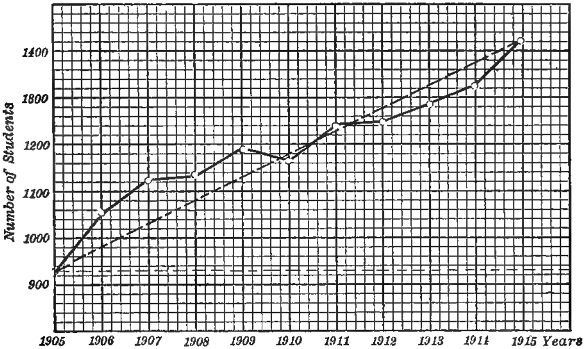
is the *average speed* of the car during this interval. Does this suggest a method for computing approximately the speed of the car at a given instant?

10. A certain function y has the value 0, when the variable x is 0, and has the value 4, when $x = 2$. The graph of the function is a straight line. Draw the graph and tabulate, from the graph, the values of y when $x=1$, 3, 4, 5, 6. What is the algebraic relation between y and x?

11. The graph of a certain function is a straight line. Draw this graph, knowing that $y = 0$, when $x = -1$, and that $y = 4$, when $x = 3$. Discover the equation connecting y and x.

26. Statistical Graphs. One of the most generally familiar uses of the graph is in connection with the representation of statistical data. The figure below represents the enrolment in Dartmouth College during the years 1905–1915. The method of its construction should be clear without further explanation.

An essential difference between this sort of graph and those previously considered must, however, be noted. Strictly speaking, the graph consists only of the *points* forming the corners of the broken line in the figure. The dates, 1905, 1906, ··· refer to the beginning of the college year in September of the years given, and the points plotted give the enrolment at the beginning of each such year. The straight lines joining these points are drawn merely for *convenience*, as an aid to the eye in following the *changes* in the enrolment from year to year. The points of these lines between the end points have *no significance.* The range of the variable here consists of the finite number of dates, 1905, 1906, ···, 1915; and the function considered is *discontinuous.* In such a graph interpolation is obviously impossible.



Enrolment of Dartmouth College, 1905–1915

FIG. 16

QUESTIONS

(1) During what periods did the enrolment increase? decrease?

(2) What was the percentage of increase during the 11 years?

(3) What was the average rate of increase (in students per year) from 1905 to 1915?

(4) If the first point of the graph (1905) be joined to the last point (1915) by a straight line (see figure), how is the steepness of this line related to the average rate of increase?

EXERCISES

1. The maximum temperatures (in degrees Fahrenheit) at Hanover, N.H., on successive days from Oct. 1 to Oct. 15, 1914, were respectively as follows:

59.6, 74.8, 79.7, 82.1, 78.9, 66.6, 61.4, 73.7, 82.5, 73.2, 78.9, 66.8, 55.0, 57.0, 63.5.

Construct a graph representing these data by a broken line. Is interpolation possible? Why?

2. American shipping statistics give the total iron and steel tonnage built in the U.S. for the years 1900–1914 as follows:

Year	Tonnage	Year	Tonnage	Year	Tonnage
1900	196,851	1905	182,640	1910	250,624
1901	262,699	1906	297,370	1911	201,973
1902	280,362	1907	348,555	1912	135,881
1903	258,219	1908	450,017	1913	201,665
1904	241,080	1909	136,923	1914	202,549

Draw the graph. Is interpolation possible? Why?

27. Summary. As has already been sufficiently indicated, the object of our work thus far has been to make clear the concept of a ***function.*** To this end we have considered a variety of *special* functions. Confining ourselves at present to the conception of what we have had occasion to define as a *single-valued* function of *one variable,* we have seen that the essential characteristic of such a function may be defined as follows:

A variable y is said to be a ***function*** of another variable x, if when a value of x is given, the value of y is determined.

A *variable* is a quantity which throughout a given discussion assumes a number of different values. The values which a variable may assume constitute the *range* of the variable in question.

The range of a variable may be limited or not according to circumstances.

We have become acquainted with three methods of representing a function: the *analytic*, the *tabular*, and the *graphic.*

We have made a beginning in the *classification* of functions: *single-valued* and *multiple-valued* functions; *continuous* and *discontinuous* functions; *increasing* and *decreasing* functions; functions of *one* variable and of *more than one* variable.

We have had occasion to note some of the questions that may arise in the consideration of a function: To determine the value of the function when the value of the variable is given; the converse problem, to determine the value (or values) of the variable, corresponding to a given value of the function. Both of these problems may involve the process of ***interpolation.*** The *maximum* or *minimum value* of a function (and the value of the variable for which this maximum or minimum occurs) is often of importance. So also is the ***rate*** at which a function changes its values. This, we have seen, is intimately connected with the steepness of the graph of the function.

28. Algebra as a Tool. The methods to be used in the future for the study of functions and their applications group themselves naturally under three headings corresponding to the methods of representing a function: *graphs, analysis, tables.*

The first of these we have already considered. It has the advantage of presenting the variation of the function vividly to the eye; in this respect it is the superior of either the tabular or the analytic method of representation. It lacks precision, however, since any graph drawn on a piece of paper is in the nature of the case an approximation.*

The analytic representation by means of a formula we have touched only very briefly. One of its chief advantages is that of the utmost precision and conciseness. This very conciseness, however, tends to obscure the properties of the function. The tools which enable a sufficiently skillful operator to bring out the hidden properties inherent in a formula are comprised in what is known as *mathematical analysis,* of which the processes of *elementary algebra* form the foundation.

The more important functions have been tabulated. Such tables are used primarily to facilitate numerical computations. We shall have occasion to use tables frequently.

The next chapter is devoted to a brief discussion of certain algebraic processes and of their relation to the graphic representation already discussed.

QUESTIONS FOR REVIEW AND DISCUSSION

1. Give examples from your own experience of quantities that are functionally related. In each case, state as many properties of the function as you can (continuous or discontinuous, increasing or decreasing, etc.).

2. State some general laws and discuss the functional relations they illustrate.

3. Would it be desirable to define a function as follows: y is a function of x, if y changes its value whenever the value of x changes? Why?

4. Give, from your experience, concrete examples of the use of an arithmetic scale. Of an algebraic scale. What are the distinguishing characteristics of these two scales?

5. Describe the three methods of representing a function and discuss the advantages and disadvantages of each.

6. If the graph of a function y of x is a straight line, and the value of the function is known for $x = 4$ and for $x = 5$ (say these values are 20 and 26, respectively), how can the value of the function for $x = 4.5$ be *calculated* (not read from the graph)? For $x = 4.2$? For $x = 5.7$?

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* On the other hand, we can conceive, theoretically, of a graph which is entirely accurate.

MISCELLANEOUS EXERCISES

1. The following table gives the pressure of wind in pounds per square feet in terms of the velocity of the wind in miles per hour:

Miles per hour	5	10	15	20	30	40	50	60	70	80
Lb. per sq. ft.	0.1	0.5	1.1	2.0	4.4	7.9	12.3	17.7	24.1	31.5

Represent the function graphically. Determine approximately the velocity which will produce a pressure of 10 lb. per square feet. What does the increasing steepness of the curve signify?

2. The following table, prepared by the U.S. Weather Bureau, gives the average monthly values of relative humidity at the stations given:

	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
New York	75	74	71	68	72	72	74	75	76	74	75	74
Chicago	82	81	77	72	71	73	70	71	70	72	77	80
New Orleans	79	80	77	75	73	77	78	79	77	74	79	79
San Francisco	80	78	78	78	79	80	84	86	81	79	77	80

Plot on the same sheet of paper. Is interpolation possible? Why?

3. The following table gives the average weight of men and women for various heights:

HEIGHT	WEIGHT IN LB.		HEIGHT	WEIGHT IN LB.	
	Men	Women		Men	Women
5 ft.	128	115	5 ft. 8 in.	154	148
5 ft. 2 in.	131	125	5 ft. 10 in.	164	160
5 ft. 4 in.	138	135	6 ft.	175	170
5 ft. 6 in.	145	143	6 ft. 2 in.	188	

Represent the two sets of data on the same paper and draw any conclusions that seem reasonable. Is interpolation possible? Why?

4. The attendance at a base ball park on successive days was as follows: 1002, 1800, 1875, 1375, 1500, 2750, 3520. Represent these data by points in a plane. Is a curve drawn through these points of any significance? Explain your answer.

5. The London Economist gives the following table showing the net tonnage of steamships and sailing vessels on the register of Great Britain and Ireland from 1840 to 1912:

YEAR	STEAMSHIP	SAILING VESSEL	YEAR	STEAMSHIP	SAILING VESSEL
1840	87,930	2,680,330	1909	10,284,810	1,301,060
1860	454,330	4,204,360	1910	10,442,719	1,112,944
1880	2,723,470	3,851,040	1911	10,717,511	980,997
1900	7,207,610	2,096,490	1912	10,992,073	902,718

Represent these data graphically on the same sheet of paper. What fact does this graph vividly portray?

6. The temperature drop t below 212° at which water will boil at different elevations and the elevation h in feet above sea level are connected by the relation $h = t^2 + 517t$. Construct a table of values of h for $t = 0, 5, 10, 15, 20, 25, 30$, and draw the graph of h as a function of t. At what temperature will water boil on Pike's Peak, 14,000 feet above sea level? About how high is it necessary to go in order that water will boil at 200°?

CHAPTER II

ALGEBRAIC PRINCIPLES AND THEIR CONNECTION WITH GEOMETRY

29. Numbers and Measurement. We have already had occasion to distinguish between two kinds of numbers:

(*a*) Numbers each of which represents a *magnitude* only;

(*b*) Numbers each of which represents a *magnitude* and *one of two opposite senses, i.e.* the so-called *signed* numbers.

It seems desirable at this point to recall the familiar classification of these numbers and the way in which they serve to give the measures of magnitudes. We confine ourselves first to the numbers of Type (*a*) above.

INTEGERS. The first numbers used were the so-called ***whole numbers*** or ***integers,***

$$1, 2, 3, 4, \cdots,$$

which represent the results of *counting* and answer the question: *How many?* They also represent the results of measurements, when the magnitudes measured are exact multiples of the unit.

THE RATIONAL NUMBERS. When the magnitude measured is not an exact multiple of the unit of measure, other numbers called ***fractions*** must be used. These numbers are intimately associated with the idea of a *ratio.* Thus, in geometry, two line segments *AB* and *CD* are called ***commensurable,*** if there exists a third segment *PQ* of which each of the other two is an

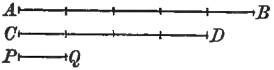

FIG. 17

exact multiple (Fig. 17). PQ is then called a *common measure* of AB and CD. If AB is exactly m times PQ and CD is exactly n times PQ, m and n being integers, we say that *the ratio of AB to CD is m/n*, and we write

$$\frac{AB}{CD}=\frac{m}{n}.$$

If CD is the unit of length, we have

$$\textit{the measure of } AB=\frac{m}{n}.$$

A number which can be written as a fraction in which the numerator and denominator are both integers is called a ***rational number***.*

Such numbers suffice to represent the measure of any magnitude which is commensurable with the unit of measure.

THE IRRATIONAL NUMBERS. If two magnitudes have no common measure, they are called ***incommensurable***. Thus we know from our study of geometry that the diagonal of a square (Fig. 18) is not commensurable with one of its sides.† Hence, the length of the diagonal of a square whose side is 1 unit cannot be expressed exactly by any rational number. To meet this deficiency the so-called ***irrational numbers***, such as the $\sqrt{2}$, were introduced.

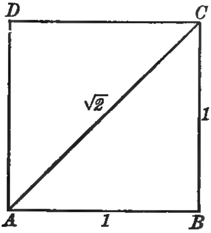

FIG. 18

It is beyond the scope of this book to treat irrational numbers fully. But we may note that they serve to express *the*

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* Observe that according to this definition the rational numbers include the integers. The number "zero" is also classed among the rational numbers. See § 30.

† If AB and AC had a common measure l, such that $AB=m\times l$ and $AC=n\times l$, where m and n are integers, it would follow that $n^2=2m^2$; but this relation cannot hold for any integers m and n. Why?

ratio of pairs of incommensurable magnitudes, and, in particular, to express *the measure of any magnitude which is incommensurable with the unit.*

Moreover, *any irrational number may be represented approximately by a rational number with an error which is as small as we please.* This follows from the following considerations.

It is important to note that *the result of any actual direct measurement is always a rational number.* For example, in measuring a distance, we use a foot rule marked into fourths, or eighths, or thirty-seconds of an inch, or else some more accurate instrument divided into hundredths or thousandths of a unit, and we always observe how many of these divisions are contained in the length to be measured. The result is, therefore, always a rational number m/n where n represents the number of parts into which the unit was divided. Any such actual measurement is, of course, an *approximation.* The greater the accuracy of the measurement (and this accuracy depends among other things on the number of divisions of the unit) the closer is the approximation. Since we may think of the unit as divided into as many divisions as we please, we may conclude that *any magnitude can be expressed by a rational number to as high a degree of accuracy as may be desired.* Thus, the length of the diagonal of a square whose side measures 1 in. is expressed approximately (in inches) by the following rational numbers: 1.4, 1.41, 1.414, 1.4142. These decimals are all rational approximations, increasing in accuracy as the number of decimal places increases, to the irrational number $\sqrt{2}$.*

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* Surds, *i.e.* indicated roots of rational numbers, are not the only irrational numbers. The familiar $\pi = 3.14159 \cdots$ is an example of an irrational number which is not expressible by means of any combination of radicals affecting rational numbers.

30. The Number System of Arithmetic. The (unsigned) rational and irrational numbers, together with the number *zero* (which is counted among the rational numbers), constitute the ***number system of arithmetic.***

31. The Number System of Algebra. Corresponding to any unsigned number a (except 0) there exist two signed numbers $+a$ and $-a$. The magnitude represented by a signed number is called the *absolute value* of the number, and is indicated by placing a vertical line on each side of the number. Thus the absolute value of $+5$ and of -5 is 5; in symbols, $|+5| = |-5| = 5$.

The signed numbers are called *rational* or *irrational* according as their absolute values are rational or irrational. The entire system of positive and negative, rational and irrational, numbers and zero* is called the *real number system* and any number of this system is called a *real* number. These numbers are contained in the so-called ***number system of algebra.***†

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* Note that zero is neither positive nor negative. It has no sign.

† The number system of algebra contains also the so-called imaginary or complex numbers, which will be discussed later. It may be noted that the words rational, irrational, real, imaginary, are here used in a technical sense. The popular meanings of the terms have no significance. $\sqrt{2}$ is no more "irrational" (*i.e.* absurd or crazy) than the number 2; and the imaginary numbers are just as "real" in the popular use of the term as are the (technically) real numbers. Historically, the reason for the use of these words is, however, connected with their customary meaning. For, while the integers and rational numbers are of great antiquity, the irrational numbers were not introduced until about the fifteenth century A.D., although incommensurable ratios were discussed by the ancient Greeks. At that time their nature was not thoroughly understood, and it was not unnatural then to designate them as irrational. Similar remarks could be made about the introduction of the imaginary numbers toward the end of the eighteenth century. We may add that what we now call "negative" numbers were in the fifteenth century often referred to as "fictitious numbers."

32. Geometric Representation. Coördinates on a Line. It follows from § 29 that the rational and irrational numbers are just sufficient to express the length of any line segment. Every segment on a line having one extremity at a given point or *origin* O can be represented by such a number; and every such number will determine a definite one of these segments, the unit of measure having been previously chosen.

This leads at once to the idea of an *arithmetic scale*, if we confine ourselves to the numbers of arithmetic, and to the idea of an *algebraic scale*, if we choose one of the directions on the line to be positive, and use the real numbers of algebra to represent the (now) *directed segments*. In the future we shall generally confine our discussion to the algebraic case. No confusion need arise from regarding an arithmetic scale as the positive half of an algebraic scale, nor from regarding the numbers of arithmetic as equivalent to the positive numbers (and zero) of the real number system.*

It is often convenient to regard the number x which originally represented the length and the direction from O to a point P of the line as representing the point P itself, in which case we call x the ***coördinate*** of P (Fig. 19). When we have chosen a point O as origin, selected a unit of length, and specified which of the two directions on the line is positive, we say that we have established a ***system of coördinates*** on the line. When this has been done, every point P of the line is represented by a number, and every real number represents a definite point of the line.

FIG. 19

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* For this reason we shall often omit the + sign in writing a positive number; *e.g.* write simply 5 for +5. The context will always tell whether the number in question is signed or not.

33. Coördinates in a Plane. We may now give the precise mathematical formulation of the process already used (in connection with the construction of the graphs of functions) for "plotting" points in a plane. The essential features of this process are as follows (Fig. 20). We locate arbitrarily in the plane two algebraic scales, a horizontal one called the ***x-axis***, and a vertical one called the *y-axis*. These two scales, called the ***axes of reference***, intersect in the zero point of each scale; this point is called the ***origin***. The position of any point P in the plane is then completely determined if its distance and direction from each of these axes is known. The units on the two scales are arbitrary; they may or may not be equal to each other. The distance from either axis must, however, be measured in terms of the unit of the other axis, *i.e.* of the axis parallel to which the measurement takes place. Thus, in Fig. 20, where the units on the axes are different, the point P_1 is determined by the distance $x = 3$ units from the y-axis (meas-

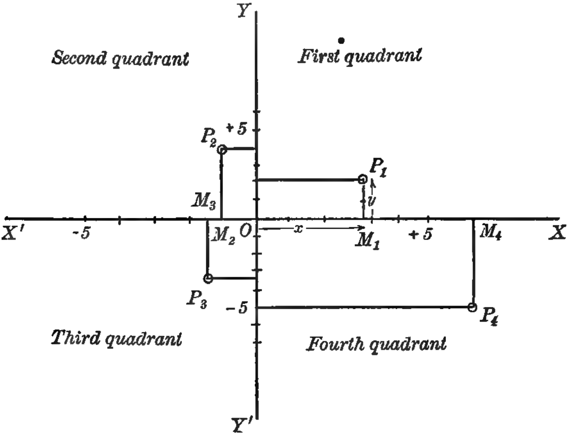

FIG. 20

ured in terms of the x-unit) and the distance $y = 2$ units from the x-axis (measured in terms of the y-unit). Similarly, the points P_2, P_3, P_4 are determined respectively by the *directed segments* OM_2 and M_2P_2, OM_3 and M_3P_3, OM_4 and M_4P_4; the numbers representing these directed segments are *signed numbers,* so that the number gives both the magnitude and the direction of the segment. In such a *system of rectangular coördinates in a plane,* unless specifically agreed on otherwise, the positive direction on the x-axis is always *to the right;* on the y-axis, always *upward.*

We see, then, that every point in the plane is determined uniquely by a pair of numbers, and, conversely, that every pair of (real) numbers determines uniquely a point in the plane. The two numbers thus associated with any point in the plane are called the *coördinates* of the point; the number x (giving the distance and direction from the y-axis) is called the ***x-coördinate*** or the ***abscissa*** of the point, the number y (giving the distance and direction from the x-axis) is called the ***y-coördinate*** or the ***ordinate*** of the point. Any point P in the plane may then be represented by a symbol (x, y), where the abscissa of P is written first in the symbol and the ordinate of P is written last. Thus we may write (Fig. 20) $P_1 = (3, 2)$ $P_2 = (-1, 4)$, $P_3 = (-\sqrt{2}, -3\frac{1}{3})$, $P_4 = (?, ?)$.

The two axes divide the plane into four regions called ***quadrants,*** numbered as in the figure. The quadrant in which a point lies is completely determined by the signs of the coördinates of the point. Thus, the first quadrant is characterized by coördinates $(+, +)$, the second quadrant by $(-, +)$, the third by $(-, -)$, and the fourth by $(+, -)$.

34. Relations between Numbers. If two numbers a and b represent two points A and B respectively on an algebraic scale, we say that a is *less than* b (in symbols, $a < b$), if a is to the left of b, the scale being horizontal and the positive direction being to the right.* The following obvious relations are fundamental:

(1) If $a \neq b$, then either $a < b$, or $b < a$.

(2) If $a < b$ and $b < c$, then $a < c$.

EXERCISES

1. Is the date 1916 a signed number? (Does it represent simply a duration of time or does it represent a time *after* some arbitrary fixed time?) Would it be proper to represent the year 50 A.D. by $+50$ and the year 50 B.C. by -50?

2. When we designate the time of day as "two o'clock," is "two" a signed number?

3. Are the (unsigned) integers used for any other purposes than to express the result of counting or measuring? (House numbers, catalog numbers, $\cdots$)

4. State some theorems of geometry concerning ratios.

5. Find a rational approximation of $\sqrt{3}$ accurate to within 0.001.

6. Why is any actual measurement necessarily an approximation?

7. Why is it incorrect to define a rational number as one "which does not contain radicals?"

8. Why should irrational numbers be used at all, if it is possible to represent any such number by a rational number to as high a degree of approximation as may be desired?

9. Explain *from the definition of ratio* why $\frac{3}{5}$ in. and $\frac{6}{10}$ in. represent the same magnitude. Why m/n in. and pm/pn in. represent the same magnitude.

10. Two segments measure $\frac{3}{4}$ in. and $\frac{5}{2}$ in., respectively. Show that the ratio of the first to the second *according to the definition* is $\frac{3}{10}$. (Observe that $\frac{1}{4}$ in. is a common measure of the two segments.)

11. Two segments measure m/n and p/q in. respectively. Prove that the ratio of the first to the second is mq/np. (Find a common measure of the two segments.)

12. Given that $|a| < |b|$, can we conclude that $a < b$? Why? Given that $|a| > |b|$, can we conclude that $a > b$? Why?

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* Likewise, a is *greater than* b (in symbols, $a > b$), if A is to the right of B. Obviously, if $a < b$, then $b > a$.

13. Which is the greater, -3 or -4? -3.1 or $-\pi$?

14. Locate on a line the points whose coördinates are $2, -\frac{1}{2}, \frac{3}{2}, -2, 5$. What is the distance between the last two? What signed number represents the directed segment from the point $+5$ to the point -2?

15. Locate in a plane the points $(2, 3)$, $(-2, 3)$, $(2, -3)$, $(-2, -3)$, referred to a system of rectangular coördinates, the units on the two axes being equal.

16. If the abscissa of a point is positive and its ordinate is negative, in what quadrant is the point? If abscissa and ordinate are both negative?

17. If the abscissa of a point in a plane is $+2$, where is the point? If the ordinate is zero? What characterizes the coördinates of a point on the y-axis? On the x-axis? What are the coördinates of the origin?

18. The units on the two scales being equal, what is the distance of the point $(3, 4)$ from the origin? Of the point $(-1, 7)$? Of the point $(2, -1)$? Of the point (a, b)?

35. The Fundamental Operations. We shall now take up briefly the fundamental operations of addition, multiplication, subtraction, and division, and develop certain geometric interpretations and applications connected with these operations, which are of importance in what follows.

ADDITION. We note first that the operation of addition for signed numbers has an essentially different meaning from that for unsigned numbers. The addition of two unsigned numbers expresses simply the addition of magnitudes. Thus, any two magnitudes may be represented geometrically by the lengths of two line segments. The segment, whose length represents their sum, is obtained by simply placing the segments end to end to form a single segment. (Compare the process of graphic addition described in Ex. 3, p. 7.)

A signed number, on the other hand, represents a direction as well as a magnitude; it is represented geometrically by a *directed* segment. Consider two signed numbers a and b. They will be represented by two directed segments whose lengths are $|a|$ and $|b|$, respectively, and whose directions are the same or opposite according as the numbers have the same or opposite signs. Figure 21 represents the four possible cases.

The sum $a+b$ is represented by a directed segment which expresses the net result of *moving* in the direction represented by a through a distance equal to $|a|$, and *then* moving in the direction of b through a distance equal to $|b|$. The segment representing $a+b$ is the segment from the initial point of these motions to the terminal point. (See Fig. 21.)

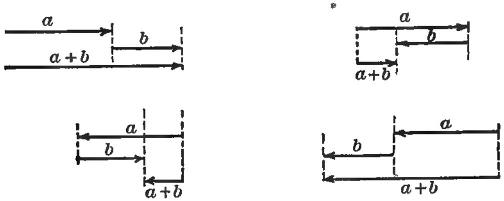

FIG. 21

The difference in the meaning of addition in the case of unsigned and signed numbers is clearly brought out by considering a simple concrete example: Suppose you walk to a place five miles distant and back again. The total distance you have walked is $5+5=10$ miles. These are unsigned numbers. On the other hand, if you represent the trip out by $+5$ and the trip back by -5, which is entirely proper, the sum $(+5)+(-5)$, which is equal to 0, does not represent the distance walked at all, but does represent the net result of your walk measured from your *starting point.* The total distance walked is represented by $|+5|+|-5|$.

It should be noted that the absolute value of the sum of two numbers *is not,* in general, equal to the sum of their absolute values. In fact all we can say in general on this point is that

(1) $$|a+b| \leqq |a|+|b|.^*$$

The equality sign holds only when a and b have the same sign.

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* The symbol $\leqq$ is read "is equal to or less than."

The geometric interpretations on the algebraic scale of adding a number x to all the numbers of the scale consists of sliding the whole scale to the right or left, according as x is positive or negative, through a distance equal to $|x|$. Figure 22 illustrates this operation for the value $x = -2$.

Every number in the upper scale is the result of adding -2 to the number below it in the lower scale. Two important consequences follow from this interpretation:

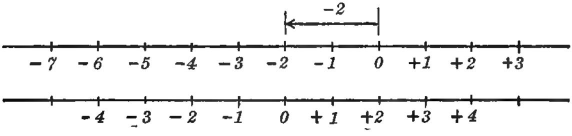

FIG. 22

(1) *If $a < b$ and x is any (real) number, then $a + x < b + x$.*

(2) *If a point P whose coördinate on a line is x is moved on the line through a distance and in a direction given by the number h, the coördinate x' of its new position is given by the relation*

$$x' = x + h. \tag{2}$$

An immediate consequence of the meaning of addition in the case of directed segments is as follows. If A, B, C are any three points on a line, then

$$AB + BC = AC. \tag{3}$$

This relation holds no matter what the order of the points on the line may be. In fact it is obvious that to move on a line from A to B and then to move from B to C is equivalent to moving directly from A to C, no matter how the points are situated on the line. As a special case of this relation we have

$$AB + BA = 0, \quad \text{or} \quad AB = -BA.$$

Multiplication. The product ab of two signed numbers a and b is defined as follows:

(1) $|ab| = |a| \cdot |b|$.

(2) *The sign of ab is positive or negative according as the signs of a and b are the same or opposite.*

The statement (2) involves the familiar ***law of signs:***

$$(+)(+)=(+), \qquad (+)(-)=(-)(+)=(-), \qquad (-)(-)=(+).$$

Geometrically, multiplication by a positive number x is equivalent to a uniform *expansion* or *contraction* of the scale away from or toward the origin in the ratio $|x|:1$, according as $|x|$ is greater than or less than 1.

This statement will become clear on inspection of the following figure (Fig. 23) which gives the construction for the multiplication of every number on the scale by x. In the first figure x has been taken equal to $+2$, in the second equal to $+\frac{1}{2}$.

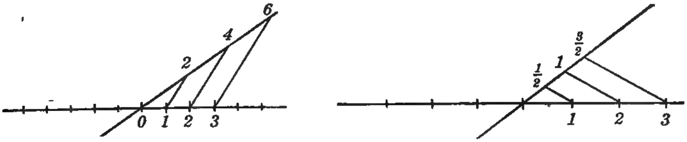

Fig. 23

The geometric interpretation of multiplication by a negative number x consists of a similar *expansion* or *contraction* in the ratio $|x|:1$ combined with a *rotation* of the whole scale about the origin through an angle of 180°. For such a rotation will change each positive number into the corresponding negative number, and vice versa, which the law of signs requires.

Here again we may note two consequences of importance:

1. *If $a < b$ and x is any (real) number, ax is less than, equal to, or greater than bx, according as x is positive, zero, or negative.*

2. *If a scale is uniformly stretched (or contracted), the origin*

remaining fixed, in such a way that the point 1 moves to the point whose coördinate is a, then the point whose coördinate is x will move to the point whose coördinate is

$$x' = ax. \tag{4}$$

Subtraction. To subtract a number b from a number a *means* to find a number x such that $x + b = a$. We then write $x = a - b$.

Such a number x can always be found. Representing a and b by directed segments having the same initial point, the meaning of addition tells us at once that the segment from the terminal point of b to the terminal point of a represents the number x sought. (See Fig. 24.) This shows, moreover, that *to subtract a number b is equivalent to adding the number* $-b$.*

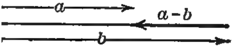

Fig. 24

Division. To *divide* a number a by a number b *means* to find a number x such that $bx = a$. We then write $x = a/b$.

It is always possible to find such a number x, *except when the divisor b is zero.* For we need merely reverse the construction given for multiplication (Fig. 23) as indicated in Fig. 25, first drawing the line joining b on the original scale to the point a on the multiplied scale and locating the required point x on the multiplied scale by a line through 1 on the original scale, parallel to the line ab.

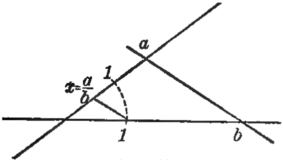

Fig. 25

In particular, we can always find a number x such that

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* It may be of interest to recall here the fact that historically the negative numbers were introduced in order to make the operation of subtraction always possible (*i.e.* even in the case when the subtrahend is greater than the minuend). But from what has just been said it appears that the device adopted for rendering the operation of subtraction more useful and convenient had the additional effect of making this operation unnecessary.

$bx = 1$, *if* $b \neq 0$. This number $1/b$ is called the *reciprocal of* b. Hence, *to divide by* b $(b \neq 0)$ *is equivalent to multiplying by* $1/b$.

THE CASE $b = 0$. This case demands careful attention. Since $0 \cdot x = 0$ for every number x, it follows that *the relation* $0 \cdot x = a$ *cannot be satisfied by any value of* x, *unless* a *is also* 0; *and will be satisfied by every value of* x, *if* a *is* 0. Hence, by the definition of division, the indicated quotient

$$x = \frac{a}{0}$$

has no meaning whatever when $a \neq 0$, and *no definite meaning* even when $a = 0$. Hence, we conclude that ***division by zero, being either impossible or useless, is excluded from the legitimate operations of arithmetic and algebra.***

36. The Function a/x. The Symbol ∞. Whereas we have just seen that division by zero is not a legitimate operation, it is highly important for us to note what happens to the fraction a/x when x assumes values *approaching* nearer and nearer to zero; as long as x does not *equal* zero, the indicated division is possible. We wish then to consider the *function* $a/x = y$ for values of x near 0. A table of corresponding values of x and y is as follows:

x	4	3	2	1	$\frac{1}{2}$	$\frac{1}{4}$	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$
$y = \frac{a}{x}$	$\frac{1}{4}a$	$\frac{1}{3}a$	$\frac{1}{2}a$	a	$2a$	$4a$	$-\frac{1}{4}a$	$-\frac{1}{3}a$	$-\frac{1}{2}a$	$-a$	$-2a$	$-4a$

Plotting the points $(x, a/x)$ with reference to two rectangular axes we obtain Fig. 26, where we have assumed a to be positive and have chosen the unit on the x-axis to be a times the unit on the y-axis.

An inspection of the table and the graph shows us that as x decreases in absolute value, a/x increases in absolute value;

more precisely, ***by choosing x sufficiently small in absolute value, a/x can be made as large in absolute value as we please.*** Further, when $x = 0$ the expression a/x has no meaning. We say *the function is not defined for the value* $x = 0$; or, *the range of the variable of this function does not include the value* $x = 0$.

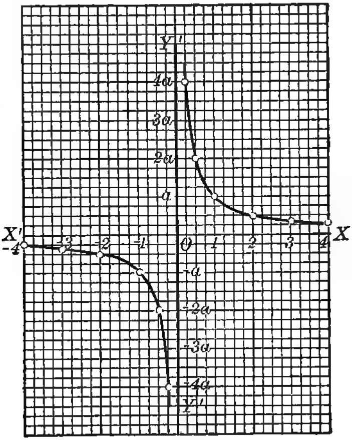


FIG. 26

The sentence expressed in black-faced italics above is sometimes written in a species of shorthand:

$$\frac{a}{0} = \infty.$$

This *looks* like an equality involving a division by 0. But *it does not mean any such thing.* The expression $a/0$ as indicating a division by 0 has already been pronounced illegitimate. For this very reason we are at liberty to use the symbol to mean something else without danger of confusion. We accordingly use it as a short way of expressing the values of the variable a/x as x is supposed to *approach* 0. Similarly, the symbol ∞, read "infinity," does not represent a number at all, but a *variable* which *increases without limit.* The above equality is, therefore, an equality between variables, and is simply a short way of writing the phrase "as the denominator of a fraction, whose numerator is constant and different from zero, approaches zero, the value of the fraction increases without limit in absolute value." Under these circumstances, we also say "the fraction *becomes* infinite." The phrase "equals infinity," which is sometimes heard, is very misleading and its use should be strictly avoided.

Returning to our table and graph, we note also that *by assigning to x a value sufficiently large in absolute value, the value of a/x can be made in absolute value as small as we please, but not zero.* The shorthand expression of this fact is

$$\frac{a}{\infty} = 0$$

or "as the denominator becomes infinite the fraction approaches 0."

37. The Directed Segment P_1P_2. As an application of the foregoing principles we will now derive a formula which will often be used in the future. Let P_1 and P_2 be any two points on an algebraic scale, and let their coördinates be x_1 and x_2, respectively. We desire to find the number representing the directed segment P_1P_2 in direction and magnitude. By definition $x_1 = OP_1$, $x_2 = OP_2$ (Fig. 27). Now, by § 35, Eq. 3, we have

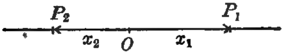


FIG. 27

$$\begin{aligned} P_1P_2 = P_1O + OP_2 &= - OP_1 + OP_2 \\ &= - x_1 + x_2, \end{aligned}$$

or, finally,

$$P_1P_2 = x_2 - x_1.$$

Thus, if $x_1 = 2$ and $x_2 = 5$, $x_2 - x_1 = +3$, and we conclude that the length of the segment P_1P_2 is 3 units and that its direction is positive (*i.e.* from left to right in the ordinary setting). On the other hand, if $x_1 = 3$ and $x_2 = -4$, we have $x_2 - x_1 = -7$, and we conclude that the length of the segment is 7 and its direction is negative (*i.e.* P_2 is to the left of P_1).

38. Concrete Illustration of the Law of Signs. The law of signs, as indeed many of the fundamental laws of algebra, is essentially a *definition*, arbitrary from a logical point of view and dictated largely on the grounds of *convenience*. The following concrete example will show how in one instance the conventions adopted in the law of signs for multiplication correspond to the concrete facts to be described.

If a train moves at a constant speed of v miles per hour, then in t hours it will travel a distance $s = vt$ miles. Here v, t, s are unsigned numbers. Now, let us change the formulation somewhat, so as to introduce the direction. At a given instant let the train be at a certain station O; let us count time from this instant ($t = 0$) so that any positive t designates an instant a certain number of hours *after* the instant $t = 0$, and a negative t designates an instant a certain number of hours *before* $t = 0$. Further, let the position of the train be determined by the signed number s representing the distance and the direction of the train from O, s being positive if the train is to the right of O (Fig. 28). Finally, let the speed and the direction in which the train is moving be given by the signed number v, v being positive if the train is moving to the right ($v = -30$, for example, would mean that the train is moving to the left at the rate of 30 miles per hour).

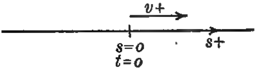


FIG. 28

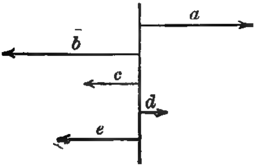
Now consider the four cases: (1) v and t both positive; (2) v positive and t negative; (3) v negative and t positive; (4) v and t both negative. Verify that the law of signs in the relation $s = vt$ gives the sign to s for which the actual position of the train in each case calls. [For example: (1) If v and t are both positive, $s = vt$ will be positive, which is as it should be. For if the train is moving to the *right*, then a certain number of hours *after* $t = 0$, when the train was at $s = 0$, it will be a certain number of miles to the *right* of O. (2) If v is positive and t negative, $s = vt$ is negative. This also is correct. For a train moving to the *right* and arriving at O when $t = 0$, was to the *left* of O at any time before $t = 0$. Etc.]

EXERCISES

1. Under what conditions is $|a+b| = |a| + |b|$?

2. Prove that if A, B, C, D, $\cdots$, L, M are any points on a line (in any order) then $AB + BC + CD + \cdots + LM = AM$.

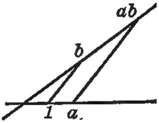
3. *Graphic Addition.* Given the directed segments, a, b, c, d, e on parallel lines (or on the same line), their sum $a + b + c + d + e$ may be found graphically as follows: On the straight edge of a piece of paper mark a point O; lay the strip along the segment a, the point O coinciding with the initial point of a; mark the terminal point of a on the paper. Then slide the paper parallel to itself so as to make it lie along b and bring the mark just made into coincidence with the initial point of b; mark the end-point of b. Then proceed similarly for the segments c, d, e. The directed segment from O to the final mark will then represent the sum sought. Why?



4. Draw directed segments representing the numbers -3, $+5$, $+2$, -6, and find their sum graphically.

5. Show how to construct a directed segment representing the product of the numbers represented by segments a and b.

[HINT. Use the adjoined figure to determine the magnitude of the product; then determine the direction. Observe that for the construction of a product we need to know the length of the unit segment, which was not necessary for a sum.]



6. Show how to construct a segment representing a/b.

7. Determine the numbers representing the directed segments from the first point of each of the following pairs of points to the second: $+8$ and $+6$, $+8$ and -6, -2 and -4, $-\frac{1}{2}$ and $+\frac{3}{2}$, $+1.4$ and -2.1, $-\frac{4}{5}$ and $-\frac{3}{2}$, $+\frac{22}{7}$ and $+3.14$.

8. By computing the numbers representing the segments, verify the relation $AB + BC = AC$, when the coördinates of A, B, C are, respectively:

(*a*) 2, 3, 4; (*b*) 2, -3, 4; (*c*) -2, 3, -4; (*d*) -2, -3, 4.

9. Find the coördinate of the mid-point of the segment joining the points whose coördinates on a line are 4 and 8; -3 and 5; -2 and -5; x_1 and x_2.

39. Insight and Technique. Most of our activities involve two more or less distinct aspects: insight and technique. On the one hand, we need to *understand* the nature of the thing we are trying to do, on the other we need *skill* in doing it. Theory and practice, planning and carrying out the plans, etc., are other ways of pointing the same distinction.

In your previous study of arithmetic and algebra the major emphasis was on the side of technique. You learned at that time *how* to carry out certain manipulations with numbers; and you gained more or less skill in using the processes. In the present course, the emphasis is to be placed on the side of insight, understanding, appreciation; the technique of algebra is to be used merely as a tool, not as an end in itself.*

40. Definitions. We propose now to recall very briefly a few of the more important conceptions and processes of algebraic technique. We shall begin with the definitions of a few terms.

When two or more numbers are added to form a *sum*, each of the numbers is called a ***term*** of the sum.

When two or more numbers are multiplied to form a *product* each of the numbers is called a ***factor*** of the product.

Any combination of figures, letters, and other symbols, *which represents a number*, is called an ***expression***. If the equality sign (=) is placed between two expressions, the result is called an ***equality***, and the two expressions are called the *members* or the *sides* of the equality.

An equality states that the two expressions represent the same number.†

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* However, we must maintain a certain amount of proficiency in the use of algebraic processes. Hence "drill exercises" will not be wholly lacking in what follows.

† Such a statement may or may not be a true statement. See § 47.

Thus, suppose a, b, c, d, p, x, y represent numbers. Then

$$a - bx + 7\,cdy = a\,(12\,y^2 - p)$$

is an equality. The left-hand member is a *sum* of three *terms;* the right-hand member consists of only one term, which is a product of two *factors.* The second term of the left-hand side is a product of two factors, while the second factor of the right-hand side is a sum of two terms.

41. General Laws of Addition and Multiplication. The following general laws we take for granted:

I. Concerning Addition:

1. *Any two numbers may be added and their sum is a definite number.*

2. *The terms of any sum may be rearranged and grouped in any way without changing the sum.*

Thus, if a, b, c, p, q represent any numbers whatever, we have, for example, $a + (b + c + p) + q = (b + q) + (c + a) + p$.

II. Concerning Multiplication:

1. *Any two numbers may be multiplied and their product is a definite number.*

2. *The factors of any product may be rearranged and grouped in any way without changing the product.*

Thus, if a, b, c, x, y represent any numbers whatever, we have, for example, $(abc)(axy) = a^2bcxy = (yx)(cba^2)$.

III. The Distributive Law: *To multiply any sum by any number m, we may multiply each term of the sum by m and add the resulting products.*

Thus, $(a + b + cd + \cdots + x)m = am + bm + cdm + \cdots + xm.$

IV. The Law of Factoring: *If every term of a sum contains the same number m as a factor, the sum contains m as a factor.*

Thus $am + bm + cdm + \cdots + xm = m\,(a + b + cd + \cdots + x).$

Observe that IV is obtained from III by simply interchanging the sides of the equality.

42. Raising to Powers. Integral Exponents. We recall also at this point the meaning and use of integral exponents. The symbol x^n, where x represents any number and n is any positive integer, is an abbreviation for the product of n factors each equal to x, *i.e.*

$$x^n = x \cdot x \cdot x \cdots \text{ to } n \text{ factors.}$$

From this definition and Principle II (§ 41) it follows at once that

$$x^m x^n = (x \cdot x \cdot x \cdots \text{ to } m \text{ factors})(x \cdot x \cdot x \cdots \text{ to } n \text{ factors})$$
$$= x \cdot x \cdot x \cdots \text{ to } m + n \text{ factors,}$$

and therefore

V*a* $$x^m x^n = x^{m+n}.$$

Similarly

V*b* $$\frac{x^m}{x^n} = x^{m-n}, \text{ if } m > n,$$

and $$\frac{x^m}{x^n} = \frac{1}{x^{n-m}}, \text{ if } n > m.$$

Also

$$(x^m)^n = x^m \cdot x^m \cdot x^m \cdots \text{ to } n \text{ factors}$$
$$= x^{m+m+m+ \cdots \text{ to } n \text{ terms}},$$

and therefore

VI $$(x^m)^n = x^{mn}.$$

Also

VII*a* $$(ab)^n = a^n b^n.$$

VII*b* $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

43. Axioms. Closely connected with Principles I, 1 and II, 1 are the familiar axioms

VIII *a* *If* $a = b$ *and* $c = d$, *then* $a + c = b + d$.

VIII *b* *If* $a = b$ *and* $c = d$, *then* $ac = bd$.

EXERCISES

1. Distinguish between insight and technique in the various professions (surgery, dentistry, engineering, etc.).

2. Complete the following propositions:

(*a*) The sum of any two integers is · · ·

(*b*) The product of any two integers is · · ·

3. What is the familiar expression in words for Principle VIII?

4. Find the results of the following indicated operations:

(1) $x^{10}x^{12}$.
(2) a^3a^5.
(3) b^nb^2.
(4) $y^{2n}y^{3n}$.
(5) $x^{n-1}x^2$.
(6) $x^{14} \div x^6$.
(7) $a^8 \div a^{10}$.
(8) $\frac{a^{2n}}{a^n}$.
(9) $(b^4)^3$.
(10) $(c^2)^6$.
(11) $(av^2)^5$.
(12) $(-c^2d^3)^5$.
(13) $(-t)^3$.
(14) $(r^ns^n)^n$.
(15) $(-x^2)^n$.

5. Multiply $x^{2a} + x^ay^b + y^{2b}$ by $x^{2a} - x^ay^b + y^{2b}$.

6. Divide $x^{5n} + y^{5n}$ by $x^n + y^n$.

7. Perform the following operations:

(1) $2^5 \cdot 2^4 =$
(2) $2^5 \cdot 4^4 =$
(3) $3^2 \cdot 2^3 =$
(4) $7^{15} \div 7^{13} =$

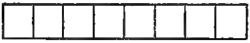
44. Discussion of Principles. In the preceding article Principles V–VII were derived from I and II, while IV is a consequence of III. We might now ask: "How do we know that Principles I, II, and III are true for all numbers?"

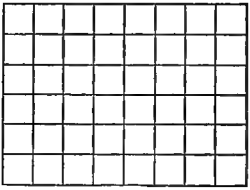
On these three principles the whole subject of algebraic technique rests. They are so simple that they may appear at first sight to be trivial. As a matter of fact their truth is by no means obvious; our unquestioned belief in them is the result of experience in using numbers. Were we to attempt a general proof, we should find it a long and difficult process which is out of place in an introductory course. Hence we simply take them for granted.

A little reflection will show that these principles are not obvious. Take for example the fact implied by Principle II, 2: *a* times *b* is equal to *b* times *a*; and let us suppose that *a* and *b* are positive integers. Now, 2 × 3 means 3 + 3 and 3 × 2 means 2 + 2 + 2. By addition we observe that the result is in both cases 6. But that simply *verifies* the general law when $a = 2$ and $b = 3$. We can thus *verify* the law in question for any two special integral values of a and b. Not only would this be extremely laborious for large values; it would still be only a verification for a *special case;* it would not be a *general proof.* Moreover, we have confined ourselves to the simplest of all numbers, the positive integers; while II, 2 asserts among other things that $ab = ba$, no matter what numbers a and b represent (rational or irrational, real or imaginary). As has been indicated in the preceding paragraph, we are not concerned in this course with proving these principles. Of great interest to us, however, are the relations existing between numbers and geometry. Accordingly we have suggested in the exercises below some geometrical interpretations of these principles which furnish intuitive proofs of certain restricted cases.

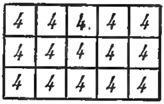
EXERCISES

1. An intuitive proof that $ab = ba$, in case a and b are positive integers: Let the integer a be represented by a group of a equal squares placed side by side so as to form a row (see the figure, where $a = 8$).





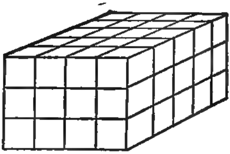
The product $b \cdot a$ is then represented by the number of squares in b such rows. Show that, if these rows be placed under each other (as in the figure, where $b = 6$), it is seen that the total number of squares is also equal to the number of squares in a columns each containing b squares. Observe that while the figure is drawn for *special* values of a and b, the argument is *general.*



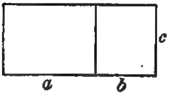
4	4	4	4	4
4	4	4	4	4
4	4	4	4	4

2. From a consideration of the adjacent figure give an intuitive proof that $5 \cdot (3 \cdot 4) = 3 \cdot (5 \cdot 4)$. Then by using the fact that $ab = ba$ show that $(3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5)$. Can this argument be made general to show $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, when a, b, c are positive integers?

3. From the adjacent figure, show how to use the idea of a rectangular pile of blocks to prove that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, when a, b, c are positive integers.



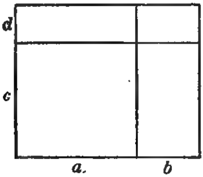
4. Assuming that the area of a rectangle is equal to the product of its base by its altitude, show that $ab = ba$, when a, b are any positive real numbers.



5. By considering the adjacent figure, interpret geometrically the relation $(a + b)c = ac + bc$.

6. Interpret geometrically the equalities

$$(a) \quad (a + b)^2 = a^2 + 2\,ab + b^2.$$
$$(b) \quad (a + b)(c + d) = ac + bc + ad + bd.$$



7. Derive the equalities in Ex. 6 from Principles I–V.

[For 6 (b), we first consider $c + d$ as a single number. III then gives $(a + b)(c + d) = a(c + d) + b\,(c + d)$. Applying II and III to each of the terms of the right-hand member of this equality, we get the desired result.]

8. Show in detail how the carrying out of the product $(ax+b)(cx+d)$ involves Principles I–V.

9. Show how these principles apply in the addition of $2\,x^2 + 7$, $3\,x + 2$, and $4\,x^2 + x + 3$.

45. Review of Algebraic Technique. We propose now to take up a few of the most elementary portions of the technique of algebra. These are all that will be needed in the immediate future. Other topics relating to technique will be recalled when they are needed.

The technique of algebra is concerned altogether with the *transforming* of expressions into other equivalent expressions which serve better the purpose in hand. The principal processes used are the following:

(*a*) ***Performing indicated operations and collecting terms.*** For example, *collect the terms in x, y, and z* in the following:

$$2x + 7y - 3z + y - 4x - 8y + 5z + 3x.$$

The result is $x + 2z$. This involves Principles I and IV.

Perform the indicated operation and collect terms in

$$(x^2 - 3x + 4)(x - 2).$$

The result is $x^3 - 5x^2 + 10x - 8$. This involves Principles I–V.

(*b*) ***The use of special products.*** The following equalities should be memorized:

$$(1)\quad (a + b)(a - b) = a^2 - b^2.$$
$$(2)\quad (a + b)^2 = a^2 + 2ab + b^2.$$
$$(3)\quad (a - b)^2 = a^2 - 2ab + b^2.$$
$$(4)\quad (x + a)(x + b) = x^2 + (a + b)x + ab.$$

(*c*) ***Factoring.*** The following cases may be specially mentioned:

i. *The difference of two squares.* Use special product (1). Thus

$$49x^6 - 4y^2 = (7x^3 + 2y)(7x^3 - 2y).$$

ii. *Trinomials of the form $x^2 + px + q$.*

Try to find two numbers whose sum is p and whose product is q, in accordance with special product (4). Thus to factor

$$x^2 - 6x - 27$$

we notice that 3 and -9 are two numbers which satisfy the requirements. Hence,

$$x^2 - 6x - 27 = (x + 3)(x - 9).$$

Again, to factor $x^2 - 10x + 25$, we notice that -5 and -5 are two numbers satisfying the required conditions. Hence

$$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2.$$

EXERCISES

1. Perform the following operations:

(*a*) $(x-4)(2x+3)$.

(*b*) $(x+a)(x+b)(x+c)$.

(*c*) $(x+h)^2-x^2$.

(*d*) $(m+n)n-(m-n)m$.

(*e*) $(a+b)^2-(a-b)^2$.

(*f*) $(a+b)^3-(a-b)^3$.

2. Factor:

(*a*) x^4-16.

(*b*) $(2a-b)^2-9(x-1)^2$.

(*c*) $ax-bx+ay-by$.

(*d*) x^4-6x^2+9.

(*e*) x^2+6x+5.

(*f*) $p^2-4p-21$.

(*g*) $t^2-(x+y)t+xy$.

(*h*) $25x^4+10x^3+x^2$.

3. Factor:

(*a*) $4a^2-5a+1$.

(*b*) $a^2+2ab+b^2-x^2$.

(*c*) $a^9-64a^3-a^6+64$.

(*d*) $1+x^2+x^4$.

(*e*) x^3-3x^2+2x.

(*f*) $2+7x-15x^2$.

(*g*) x^3+1.

(*h*) $x^4y^2-17x^2y-110$.

46. Operations with Fractions. These depend chiefly on the simple principle that the numerator and the denominator of a fraction may be multiplied by the same number (not zero) without changing the value of the fraction, and the reverse of this principle, viz. that any common factor (not zero) of the numerator and the denominator of a fraction may be removed without changing the value of the fraction.*

By means of this rule any two or more fractions can be reduced to the same denominator. The rules for adding and multiplying fractional expressions are stated symbolically as follows:

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

$$\frac{a}{b}=\frac{na}{nb}. \qquad (n \neq 0)$$

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* This principle is a direct consequence of the definition of division. Can you explain it?

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}.$$

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}.$$

Also

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}$$

and

$$-\frac{a+b}{c}=\frac{-a-b}{c}.$$

The following exercises will furnish applications of these principles.

EXERCISES

1. Express as a single fraction :

(*a*) $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$ (*d*) $\frac{a^2x}{c^2y}\cdot\frac{b^2y}{a^2z}\cdot\frac{c^2z}{b^2x}.$

(*b*) $\frac{2}{x}+\frac{3}{y}+\frac{4}{z}-\frac{7}{x+y+z}.$ (*e*) $\left(\frac{m}{n}+\frac{u}{v}\right)\left(\frac{n}{m}-\frac{v}{u}\right).$

(*c*) $\frac{r}{st}+\frac{s}{tr}+\frac{t}{rs}.$ (*f*) $\frac{1}{2}+\frac{1}{x}-\frac{3}{y}.$

Simplify the following expressions, assuming that no canceled factors have the value zero.

2. $\frac{4}{x-a}+\frac{4a}{(x-a)^2}-\frac{a^2}{(x-a)^3}.$

3. $\frac{a-c}{(a-b)(c-b)}-\frac{b+c}{(a-c)(a-b)}+\frac{a+b}{(b-c)(c-a)}.$

Ans. $\frac{2a}{(a-b)(c-b)}.$

4. $\left\{\frac{a-b}{a+b}-\frac{a+b}{a-b}\right\}\{b^2-a^2\}.$ Ans. $4ab.$

5. $\frac{x^4-y^4}{a^3-b^3}\cdot\frac{a^2+ab+b^2}{x^2+y^2}.$

6. $\left\{\frac{b^5}{a^5}-\frac{a^5}{b^5}\right\}\div\left\{\frac{b^5}{a^5}+\frac{a^5}{b^5}\right\}.$

7. $\dfrac{x^2-2xy}{xy+4y^2} \div \dfrac{x^2-4y^2}{x^2+4xy}$. Ans. $\dfrac{x^2}{y(x+2y)}$.

8. $\left(\dfrac{a}{bc}-\dfrac{b}{ac}-\dfrac{c}{ab}-\dfrac{2}{a}\right) \div \left(\dfrac{1}{1-\dfrac{2c}{a+b+c}}\right)$.

9. $\left(\dfrac{a^2-b^2}{a^3+b^3}\right)\left(\dfrac{a^2+b^2}{b}-a\right) \div \left(\dfrac{1}{b}-\dfrac{1}{a}\right)$. Ans. a.

10. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

11. $\dfrac{\dfrac{1}{y}}{1+\dfrac{x^2}{y^2}}+\dfrac{1}{2}\left(\dfrac{1}{x-y}-\dfrac{1}{x+y}\right)$.

12. $\dfrac{\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}}{\left\{1+\dfrac{c}{a+b}\right\}\left\{\dfrac{(a+b)^3-c^3}{(a+b)^3}\right\}}$. Ans. 1.

13. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{c}{a}+\dfrac{a}{b}+\dfrac{b}{c}}$.

14. $\dfrac{a^3+b^3}{a^2+ab+b^2}\left(1+\dfrac{b}{a-b}\right)\dfrac{a^3-b^3}{a^2-ab+b^2}$.

15. $\dfrac{\left\{x^2+\dfrac{y^4}{x^2-y^2}\right\}\{x^2+y^2\}}{\dfrac{x}{x+y}+\dfrac{y}{x-y}}$. Ans. $x^4-x^2y^2+y^4$.

16. If a, b, and x are positive, which is the greater,

$$\frac{a}{b} \text{ or } \frac{(a+x)}{(b+x)}?$$

Distinguish two cases $a>b$, $a<b$.

17. If $ad<cb$, then is it true for all values of the letters involved that $a/b<c/d$? Why?

47. Identities and Equations. We must recall here a vital distinction between two kinds of equalities. An equality which is true for all values of the letters (or other symbols) involved, for which both members of the equality have a meaning, is called an ***unconditional equality*** or an ***identity***. An equality which may be true for certain values of the letters involved, but is not true for all, is called a ***conditional equality*** or an ***equation***. For example, the equality $a^2-b^2=(a-b)(a+b)$ is an identity since the two members of the equality represent the same number for all values of a and b. Also the equality

$$\frac{a^2-b^2}{a-b}=a+b$$

is an identity, even though it becomes meaningless when $a=b$. Why? On the other hand $2x-8=0$ is an equation since it is true only for $x=4$. $\sqrt{x+1}=-1$ is also an equation, but it is not true for any value of x. Why?*

To *solve* an equation is to find the values of the letters for which it is true. Thus in the first example above, $x=4$ is the solution or *root* of the equation $2x-8=0$. The second equation above has no root.

48. The Relation $A \cdot B=0$. In the solution of equations the following principle is of frequent application. If a product of expressions each representing a number is zero, we may conclude that some one of the factors is zero. In the simplest case this means that if A and B represent expressions and if $A \cdot B=0$, we conclude that either $A=0$ or $B=0$.†

---

* By $\sqrt{a}$ is meant the positive square root of a.

† We must, of course, be careful to assure ourselves that each of the expressions involved represents a number for the values under consideration. Thus we cannot conclude from the relation $x \cdot (1/x)=0$, that either $x=0$ or $1/x=0$, for when $x=0$, $1/x$ is meaningless. In fact the given relation is impossible; the equality is not true for any value of x.

We may apply this principle to show the absurdity of some mistakes that are often made by the careless student. For example, a favorite mistake is to "cancel" the x in the expression

$$\frac{a+x}{b+x}.$$

This would be justified if the equality

$$\frac{a+x}{b+x}=\frac{a}{b} \tag{4}$$

were an identity. If we clear this equality of fractions by multiplying both members by $(b+x)\,b$, we obtain

$$ba+bx=ab+ax,$$

or

$$bx=ax;$$

or, finally,

$$(b-a)\,x=0.$$

Hence we conclude that equality (4) cannot be true, unless either $b=a$, or $x=0$. The "canceling operation" mentioned above is therefore unjustified.

EXERCISES

1. Treat similarly the following equalities to determine under what conditions they are true. Each one is related to an error that is sometimes made.

(*a*) Is $\sqrt{a^2+b^2}=a+b$? (Square both sides.)

(*b*) Is $\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$?

(*c*) Is $\frac{2a+b}{2c+d}=\frac{a+b}{c+d}$?

(*d*) Is $(x+y)^2=x^2+y^2$?

(*e*) Given $x^2=2x$. Are we justified in concluding that $x=2$?

2. In each of the following equalities, assuming that the letters represent real numbers, determine which are identities and which are equations. Among the latter, distinguish those that are not true for any (real) values of the letters involved; and for the others determine in their simplest form the conditions which they imply on the letters involved.

(*a*) $x^4 - y^4 = (x + y)(x - y)(x^2 + y^2)$.

(*b*) $x^2 - 3x + 2 = 0$.

(*c*) $x + \frac{1}{x} = 0$.

(*d*) $ac - bc + ad - bd = 0$.

3. Find and discuss the error in the following reasoning:

Let $x = 2$. Then $x^2 = 2x$, and $x^2 - 4 = 2x - 4$. This is equivalent to

$$(x + 2)(x - 2) = 2(x - 2).$$

Dividing both sides by $x - 2$, we get

$$x + 2 = 2.$$

But $x = 2$; hence

$$2 + 2 = 2$$

or

$$4 = 2.$$

4. Find and discuss the error in the following reasoning: Let a and b represent two numbers. Then

$$a^2 - 2ab + b^2 = b^2 - 2ab + a^2,$$

or

$$(a - b)^2 = (b - a)^2,$$

or

$$a - b = b - a;$$

hence

$$a = b.$$

PART II. ELEMENTARY FUNCTIONS

CHAPTER III

THE LINEAR FUNCTION. THE STRAIGHT LINE

49. A Linear Function. *Distance traversed at uniform speed.*

EXAMPLE. A railroad train starts 10 miles east of Buffalo and travels east at the rate of 30 miles per hour. How far from Buffalo is the train at the end of x hours?

In x hours the train travels $30x$ miles. If its distance from Buffalo is denoted by y, we have $y = 30x + 10$. Pairs of values of x and y obtained from this equation are shown in the following table.

x	0	1	2	3	4	etc.
y	10	40	70	100	130	etc.

GEOMETRIC REPRESENTATION. Let us plot as points in a plane these corresponding values of x and y. We then obtain the first of the following figures (Fig. 29). It will be noticed

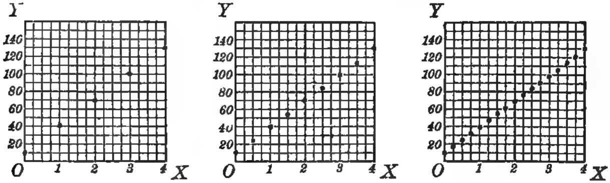

FIG. 29

that the five points appear to lie on a straight line. We have, for intermediate values of x, the values of y shown in the following table.

x	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$
y	25	55	85	115

Plotting these points we obtain the second of the above figures, in which the nine points appear to lie on a straight line. Let us calculate the value of y for some more intermediate values of x thus:

x	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{7}{4}$
y	$17\frac{1}{2}$	$32\frac{1}{2}$	$47\frac{1}{2}$	$62\frac{1}{2}$

In the third figure we see that these new points still appear to lie on the same straight line.

These considerations suggest that if we could calculate the values of y corresponding to *all* the values of x between $x = 0$ and $x = 4$, the points whose coördinates are (x, y) would all lie on a straight line joining the points $(0, 10)$ and $(4, 130)$, and would constitute the whole of this line-segment. A proof that this is the case is as follows: In Fig. 30 we have drawn the straight line joining the points A $(0, 10)$ and B $(4, 130)$. Let $(x_1, y_1)(x_1 > 0)$ be any pair of corresponding values of x and y for the function $y = 30x + 10$; we then have

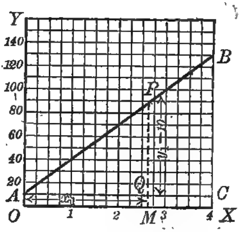

FIG. 30

(1) $$y_1 = 30x_1 + 10.$$

We now wish to prove that the point P (x_1, y_1) is on this line AB.* To do this we construct the triangles APQ and ABC by drawing lines through A, P, and B parallel to the axes. If P is on the line AB, then these triangles are similar, and if P is not on the line AB, then the triangles are not similar. Why? If the triangles are similar, $QP/CB = AQ/AC$; and conversely, if $QP/CB = AQ/AC$, the two triangles are similar. Expressed in terms of the coördinates of A, B, and P, this proportion becomes (see figure)

$$\frac{y_1 - 10}{120} = \frac{x_1}{4}{}^{**};$$

or

$$\frac{y_1 - 10}{x_1} = \frac{120}{4} = 30.$$

But from (1) we have $y_1 - 10 = 30\,x_1$ and hence

$$\frac{y_1 - 10}{x_1} = 30.$$

This proves that ***every*** *point whose coördinates* (x_1, y_1) *satisfy the relation* $y = 30\,x + 10$ *is on the straight line* AB.

Conversely, ***every*** *point on the straight line* AB *has coördinates* (x_1, y_1) *which satisfy the relation* $y = 30\,x + 10$.

For, from the figure, we have

$$\frac{y_1 - 10}{x_1} = \frac{120}{4}$$

whence

$$y_1 = 30\,x_1 + 10.$$

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* Extended beyond B, if $x_1 > 4$.

** Observe that QP and CB are measured in different units from AQ and AC. But the ratio of two line-segments is independent of the unit in which they are measured.

The straight line AB (extended indefinitely beyond B) then gives a complete representation of the function $30x + 10$, at least for positive values of x (negative values of x have no meaning in this problem). *Every pair of corresponding values of x and y gives rise to a point on AB, and every point of AB has coördinates which are corresponding values of x and y.* By virtue of this fact the line AB is called the *graph of the function* $30x + 10$, or *the locus of the equation* $y = 30x + 10$ referred to rectangular coördinates; whereas the equation $y = 30x + 10$ is called *the equation of the line AB.*

Uses of the Graph. The graph just discussed exhibits vividly to the eye several properties of the function $30x + 10$.

(1) The function steadily increases as x increases. This corresponds to the fact that the longer the train moves eastward, the greater is its distance from Buffalo.

(2) Corresponding to every positive value of x, there is a unique value of y. From the graph find y when x is 4.

(3) Corresponding to every positive value of y (greater than 10) there is a unique value of x. What is the value of x when y is 160?

(4) The last consideration means that x is also a function of y. Explicitly we have

$$y = 30x + 10,$$

whence

$$y - 10 = 30x$$

and

$$x = \frac{y - 10}{30}.$$

It is left as an exercise to draw the graph of the function

$$x = \frac{y - 10}{30}$$

by assigning values to y and computing the corresponding values of x. Compare the result with the graph in Fig. 30.

RATE OF CHANGE OF A FUNCTION. Before leaving this special case to consider a more general problem, we shall use it to illustrate a very important conception connected with a function. We have noted that when $x = 0$, $y = 10$. Starting from this initial value, as x increases from the value 0, the value of the function, *i.e.* y, changes (in this case increases). It is often of the greatest importance to know how the increase in the function y is related to the increase in x. As x increases from 0 to 1, y increases from 10 to 40; *i.e.* a change in x of one unit produces a change in y of $40 - 10$ or 30 units. The relative change is then $\frac{30}{1}$, or 30. As x increases from $x = 0$ to $x = 2$, y changes from $y = 10$ to $y = 70$, or by 60 units, and the relative change is again 30.

Let us see what the situation is in general. Let x_1 be any particular value of x and y_1 the corresponding value of y; then suppose that x_2 is any other (subsequent) value of x and y_2 the corresponding value of y. The change in x is evidently $x_2 - x_1$ and the corresponding change in y is $y_2 - y_1$. We seek the value of the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

We have from the data of the problem

$$y_2 = 30\,x_2 + 10$$

and

$$y_1 = 30\,x_1 + 10.$$

Subtracting we get

$$y_2 - y_1 = 30(x_2 - x_1)$$

and hence

$$\frac{y_2 - y_1}{x_2 - x_1} = 30.$$

We see then that the ratio of a change in the function $30\,x + 10$ to the corresponding change in x is constant and is equal to the speed of the train. We shall see presently that in any function of the first degree in x, the ratio of a change in the function to the corresponding change in x is constant.

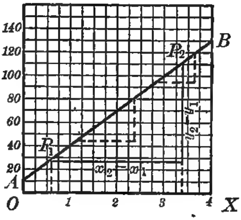

FIG. 31

Geometrically this result expresses the familiar proportionality of homologous sides of similar triangles. By reference to Fig. 31 we may readily verify that the terms $y_2 - y_1$ and $x_2 - x_1$ represent the vertical and the horizontal sides of a right triangle whose hypotenuse is on the line AB. The fact that the ratio $(y_2 - y_1)/(x_2 - x_1)$ is constant, *i.e.* always equal to 30, simply corresponds to the obvious fact that any two such triangles, no matter at what place they are drawn, or how long their sides are taken, are similar.

50. Change Ratio. The ratio

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

is called the ***change ratio*** (or sometimes the difference ratio) of the function. The difference $x_2 - x_1$ is often denoted by Δx, and the corresponding difference $y_2 - y_1$ by Δy.* The change ratio may then be written $\Delta y/\Delta x$. Explicitly, by definition, we have the following equalities:

$$\textbf{\textit{change ratio}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{corresponding change in } x}.$$

The preceding considerations suggest the theorem:

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* Δ is a Greek capital letter corresponding to our D and called delta; it is used because d is the initial of the word "difference." "Δx," is then merely an abbreviation for "difference of the x's" or "change in x" and "Δy" for "difference of the y's" or "change in y."

If the change ratio of a function is constant, the graph of the function is a straight line; and conversely.

The truth of this theorem is already sufficiently indicated in case $y_2 - y_1$ and $x_2 - x_1$ are both positive numbers. In formulating a general proof we must keep in mind that $y_2 - y_1$ and $x_2 - x_1$ may be either positive or negative and that these differences represent *directed segments.* The proof of the theorem in general will appear presently.

EXERCISES

1. Discuss fully the graph of the function $y = 2x + 3$. Prove that the graph is a straight line. Express x as a function of y. Find the change ratio and show that it is constant.

2. Proceed as in Ex. 1 for each of the functions:
(*a*) $5x + 2$, (*b*) $x + 12$, (*c*) $3.2x + 8.4$.

3. Prove that the change ratio for the function $y = mx + b$ is m.

4. A steamer 150 miles east of Toledo starts to travel west at a uniform rate of 15 miles per hour. Express its distance y east of Toledo at the end of x hours. Draw the graph of the function and prove that it is a straight line. Does the distance y increase as x increases? Calculate the change ratio and show that it is constant. What is the significance of the negative sign? At what time is the steamer 10 miles east of Toledo? When does it reach Toledo? How are the last two results shown in the graph? What is the significance of the graph that extends below the x-axis?

5. Give examples, drawn from your experience, of functions which
(*a*) increase as the variable increases;
(*b*) decrease as the variable decreases.

6. Consider the function $y = x^2$. Calculate the corresponding values of y when $x = 0, 1, 2, 3, 4, 5$. Plot the corresponding points and observe that they are *not* on a straight line. Calculate the change ratio of this function for $x = 0$ and $\Delta x = 1, 2, 3$, and observe that it is *not* constant.

7. The cost of printing certain circulars is computed according to the following rule. The cost for the first one hundred circulars is $2 and for each succeeding one hundred $0.50. Express the cost y in dollars of x hundred circulars. Draw the graph of the function and determine from the graph the cost of printing 475 circulars. What does the change ratio of the function y express in this case? *Ans.* $y = \frac{1}{2}x + \frac{3}{2}$.

51. The General Linear Function $mx + b$. *The change ratio of every function of the form $mx + b$ is constant.*

PROOF. Let (x_1, y_1) and (x_2, y_2) be any two pairs of corresponding values. Then

$$y_1 = mx_1 + b \text{ and } y_2 = mx_2 + b;$$

hence

$$y_2 - y_1 = m(x_2 - x_1),$$

i.e.

$$\frac{y_2 - y_1}{x_2 - x_1} = m.$$

Conversely, *if the change ratio of the function y of x is constant and equal to m, the function has the form $y = mx + b$.* Let (x_1, y_1) be a particular pair of corresponding values and (x, y) *any* other pair of corresponding values. By hypothesis the change ratio is equal to m; *i.e.*

$$\frac{y - y_1}{x - x_1} = m$$

or

$$y = mx - mx_1 + y_1;$$

but $-mx_1 + y_1$ is a constant, say b. Hence

$$y = mx + b.$$

Hooke's law affords an excellent illustration of the above theorem. This law states that the length y of a piece of wire under tension is equal to its original length b, plus the stretch, which is proportional to the force x causing it. Thus, $y = b + mx$.

This law may also be stated simply by saying that the change ratio of the length y, with respect to the pull x, is constant.

The preceding considerations lead to the following theorem.

THEOREM. *If a function y of a variable x is such that any change in the value of the function is always equal to m times the corresponding change in the variable, the function y is given by a relation of the form $y = mx + b$; and, conversely, in any function of this form any change in y is always m times the corresponding change in x.*

52. The Graph of a Linear Function. Let $P_1(x_1, y_1)$ be any point on the graph of a linear function (Fig. 32). From P_1 draw to the right a positive horizontal segment P_1Q_2 equal in length to $x_2 - x_1$, *i.e.* Δx. Through Q_2 draw a vertical segment and let it meet the graph in the point P_2. The segment Q_2P_2 is equal to $y_2 - y_1$, *i.e.* Δy, and is positive if P_2 is above P_1 (Fig. 32 *a*) and negative if P_2 is below P_1 (Fig. 32 *b*). Now let us take another positive change $\Delta x = x_3 - x_1$, represented by P_1Q_3 and the corresponding change $\Delta y = y_3 - y_1$ represented by Q_3P_3. If the change ratio is constant, then (1) either P_2 and P_3 are both above P_1 or they are both below P_1, according as the given constant is positive or negative; and (2) the triangles $P_1Q_2P_2$ and $P_1Q_3P_3$ are similar. Therefore the points $P_1P_2P_3$ are on a straight line, if and only if the change ratio is constant.

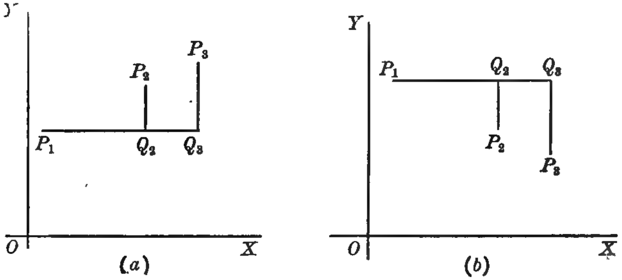

FIG. 32

THEOREM. *The graph of any function of the form $y = mx + b$ is a straight line.*

To draw the graph of such a function we need, therefore, merely to plot two points of the graph and draw the straight line through them.*

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* While two points are sufficient to determine the line completely, it is desirable to find a third point as a check on the other two. Moreover, it is advisable to take the points as far apart as convenient. Why?

Example. Draw the graph $y = 3x + 2$. We notice that $(0, 2)$ and $(4, 14)$ are two points on the graph. The line joining these two points is the required line. Check by plotting a third point.

53. The Slope of a Straight Line. The graph of the function $y = mx + b$ may be obtained by observing that $x = 0$, $y = b$ and $x = 1$, $y = m + b$ are two pairs of corresponding values of x and y. In the adjoining figure (Fig. 33) we have plotted the two points $B(0, b)$ and $C(1, m + b)$ on the assumption that both of the quantities b and m are positive numbers. The change ratio, as we have seen, is m. In the figure this ratio is DC/BD.

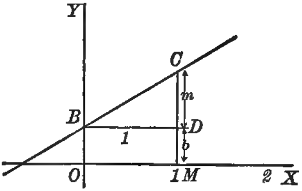


Fig. 33

Now suppose that b remains constant and that m takes on successively different values. Under the hypothesis that b and BD remain fixed, the points B and D would remain fixed and the point C would move up or down on the vertical line through D, according as m increases or decreases. The line BC would then rotate about the point B, becoming steeper if m is increased and less steep if m is decreased. The change ratio m then measures the steepness of the line. The term *change ratio* applies to the *function* $mx + b$; when applied to the straight line $y = mx + b$, it is called the ***slope*** of the straight line.

54. Remarks Concerning the Slope of a Line. We assumed in the last section that both b and m were positive numbers. Let us now suppose that b is still positive, but that m is negative. Observe that in the preceding figure $MC = MD + DC = b + m$. Recalling that the relation $MC = MD + DC$ holds universally for any three points M, D, C, on a line (Art. 35), the interpretation of a negative m, *i.e.* DC, is that the point C is below the point D. (Cf. also § 52.) A negative value of m then merely causes the line to slope downward in going from left to right, while, as we have seen, a line with a positive m slopes upward. When $m = 0$, the line is parallel to the x-axis. Indeed the equation $y = mx + b$ becomes, for the value $m = 0$, the equation $y = b$. This equation, when interpreted as a function of x, means that for every value of x, the value of y is b; the graph of such a function is obviously a straight line parallel to the x-axis. Since a change in x in this case produces no change in y, the change ratio is zero. Finally, if b is negative, nothing is changed except that the point B is below the origin O. A positive m still indicates an upward slope and a negative m a downward slope, in passing from left to right.

The number b, we have seen, represents the segment from the origin to the point in which the line cuts the y-axis. This segment is called *the y-intercept of the line.* Similarly, the segment from the origin to the point in which the line cuts the x-axis is called *the x-intercept of the line.*

We have then the following results:

The straight line represented by the equation $y = mx + b$ *has a slope equal to* m *and a y-intercept equal to* b. *In passing from left to right, the straight line slopes downward if* m *is negative and upward if* m *is positive; if* m *is zero, the line is parallel to the x-axis.*

In the terminology of functions we have:

The linear function $mx + b$ *is an increasing function of* x *(i.e. the function increases as* x *increases) if the change ratio* m *is positive, and a decreasing function of* x *(i.e. the function decreases as* x *increases) if* m *is negative. It is a constant function if* m *is zero.*

55. Examples of Linear Functions. EXAMPLE 1. On a Fahrenheit thermometer the freezing point of water is placed at 32°, the boiling point at 212°. On a Centigrade thermometer the freezing point is at 0°, the boiling point at 100°. Express the temperature of $y°$ Fahrenheit as a function of $x°$ Centigrade.

SOLUTION: $y = 32$ when $x = 0$. Also the range of temperature from the freezing point to the boiling point of water is 212°–32° or 180° F. while it is 100° C. Therefore it follows that an increase of 1° C. is equivalent to an increase of $\frac{9}{5}$ of a degree F. Now as the temperature increases from 0° to $x°$ C. the change in the number of degrees is x. This change in temperature is equivalent to an increase from 32° to $y°$ F. The change in the number of degrees is then

$$y - 32 = \tfrac{9}{5}x, \text{ or } y = \tfrac{9}{5}x + 32.$$

As a check we may observe that, when $x = 100$, the formula gives $y = 212$, as it should. Are negative values of x admissible? Figure 34 represents the graph of this function. It was drawn by using the points A (–30, –22), B (100, 212). [Why is it desirable to choose points so far apart?] This graph may be used to read off without computation the approximate temperature in F. for a given temperature in C. For example, to $x = 22$ corresponds $y = 72$, approximately. Therefore 22° C. is equivalent to about 72° F. By computation we find that $y = 71.6$.

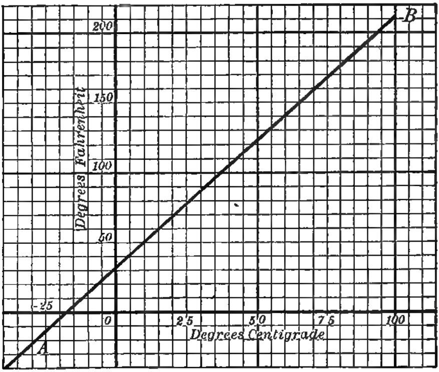


FIG. 34

Example 2. A bar of iron 3 ft. long at 60° F. will expand or contract if the temperature increases or decreases. The increase in length is proportional to the increase in temperature (physical law). More precisely, an increase of 1° F. produces an increase of 0.0000027 ft. In this case we have $m = 0.0000027$. If y represents the length at $x°$ F., we have

$$y = 3 + m(x - 60).$$

Does this relation hold also when $x < 60$? Why? Can you draw the graph?

We shall now give an example in which m is negative.

Example 3. An aëroplane starts 200 miles east of Chicago and travels towards Chicago. Express its distance y from Chicago in miles at the end of t hours, if the aëroplane moves at the rate of 82 miles per hour.

Solution: According to the data the distance from Chicago is decreasing at the rate of 82 miles per hour, *i.e.* $m = -82$. Therefore,

$$y - 200 = -82\,t, \qquad \text{or} \qquad y = -82\,t + 200.$$

Draw the graph. What is the significance of a negative y (*e.g.* when $t = 5$)? When does the aëroplane reach Chicago? When is it at a point 53 miles east of Chicago? When is it 52 miles west of Chicago? How could these questions be answered from the graph alone?

EXERCISES

1. On a Réaumur thermometer the freezing point of water is at 0°, the boiling point at 80°. Express the temperature in degrees Fahrenheit in terms of the temperature in degrees Réaumur. Draw the graph and show how it may be used.

2. Is there any temperature whose measures in the Fahrenheit and in the Centigrade scales are equal? Answer by computation. How could the result be found graphically?

3. A cistern that already contains 300 gallons of water is filled at the rate of 50 gallons per hour. Show that the amount of water y in this cistern at the end of x hours is $y = 50\,x + 300$. Draw the graph and discuss. How would the function be changed, if the cistern were being emptied at the rate of 50 gallons per hour?

4. A tank contains 16 gallons of water. A faucet is opened which admits 4 gallons per minute. Express the amount, w, of water in the tank at the end of t minutes. Draw the graph. Do negative values of t have any significance? When will the tank contain 37 gallons?

5. A tank containing 37 gallons of gasolene is emptied at the rate of 5 gallons per minute. Express the amount of gasolene in the tank at the end of t minutes. Draw the graph. When will the tank be emptied? For what range of values of t has the function any significance?

6. On a certain date a man has $5 in the bank. At the end of every week he deposits $3. How much money has he in the bank at the end of x weeks? Draw the graph of this function. How is the rate of increase shown in the graph?

7. On a certain day a man has $100 in the bank. At the end of every week he draws out $5. How much money has he in the bank at the end of x weeks? Draw the graph of this function. How is the rate of decrease shown in the graph?

8. In experiments with a pulley, the pull P in pounds required to lift a load L in pounds, was found to be $P = 0.15\,L + 2$. Plot this relation. How much is P when L is zero. How much is P when L is 10 lbs.?

9. If h represents the height in meters above sea level, and b the reading of a barometer in millimeters, it is known that $b = k + hm$, where k and m are constants. At a height of 110 meters above sea level the barometer reads 750; at a height of 770 meters it reads 695. What equation gives the relation between b and h? Draw the graph of this equation and from the graph determine h when $b = 680$.

56. Linear Interpolation. The fact that the change Δy in a *linear function* y is proportional to the change Δx in the variable x makes it possible to *interpolate* readily. For example, if we know that y is a linear function of x, and that $y = 432.50$ when $x = 32.0$ and that $y = 436.90$ when $x = 33.0$, we can calculate mentally the value of y when $x = 32.3$. For we know that in this case $\Delta y = 4.40$ when $\Delta x = 1.0$; hence $\Delta y = 4.4 \times 0.3 = 1.32$ when $\Delta x = 0.3$. Hence y is 433.82 when $x = 32.3$. This process is known as *linear interpolation.* Why would this process not apply directly to functions that are not linear?

EXERCISES

Assuming that y is a linear function in each of the following cases compute the values of y indicated.

1. When $x = 10$, $y = 50$; when $x = 14$, $y = 90$; when $x = 11$, $y = ?$

2. When $x = 2.4$, $y = 9.8$; when $x = 2.5$, $y = 8.6$; when $x = 2.42$, $y = ?$

57. Graphic Solution of Problems. Whenever we know at the outset that the solution of a problem is going to depend on the consideration of one or more linear functions, we can often solve the problem graphically without determining these linear functions analytically. Such a method is advantageous whenever the computation is difficult or tedious and when great accuracy is unnecessary. In order to decide whether the functions involved are linear or not, we usually have recourse to the theorem (§ 51) that, whenever the change in the function is proportional to the change in the variable, the function is linear. This is true, for example, in all cases of motion at a constant speed on either a straight or curved path; the distance is then a linear function of the time.

The following example will serve to illustrate the graphic method of solution.

EXAMPLE. At 7 A.M. a man starts to go up the 7-mile carriage road of Mt. Washington. At 9 o'clock he passes a party of ladies coming down. He reaches the top at 10 o'clock and, finding no view, he immediately sets out on the return trip, which takes $1\frac{3}{4}$ hrs. As he reaches the hotel from which he started he notices the party of ladies just arriving. At about what time did the ladies leave the top, assuming that the man kept up an approximately constant rate of speed on the way up and the ladies on the way down?

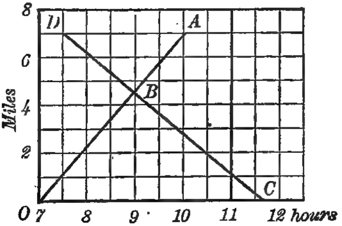


FIG. 35

To solve the problem, we represent on a horizontal axis the time, marking the hours 7, 8, 9, 10, 11, 12 and on the vertical axis the distances from the hotel at the foot of the mountain. The graph of the man going up the mountain is a straight line starting at O (at 7 A.M. he was at the hotel) and ending at a point A representing 10 o'clock and 7 miles from the hotel. Regarding the ladies, we know that the graph of their descent is also a straight line. At 9 o'clock they were the same distance from the hotel as the man. The point B on the line OA, corresponding to 9 o'clock, is then one point of the ladies' graph. Another point is the point C at 0 distance from the hotel at 11:45. The line BC is then drawn and extended to D, representing 7 miles distance from the hotel. It is seen that the ladies started at about 7:30. How far was the man from the top when he met the ladies?

58. Sum of Two or More Functions. Let $m_1x + b_1$, $m_2x + b_2$, $\cdots$, $m_kx + b_k$ be any k linear functions of x. The sum of these functions is $(m_1x + b_1) + (m_2x + b_2) + \cdots + (m_kx + b_k)$ and this is equal to

$$(m_1 + m_2 + \cdots + m_k)\,x + (b_1 + b_2 + \cdots + b_k),$$

which is again of the form $mx + b$. The result may be stated as follows: *The sum of any number of linear functions of x is itself a linear function of x.*

EXAMPLE. An empty tank is being filled by a faucet supplying 2 gallons of water per minute. After this faucet has been running 5 minutes a second faucet is turned on which supplies water at the rate of 3 gallons per minute. When the two faucets have been running together for 6 minutes, an outlet is opened, but both faucets continue to run. If the tank is empty at the end of 32 minutes, counted from the start, draw a graph representing the amount of water in the tank at any instant. Find approximately the rate of flow from the outlet, which may here be considered constant.

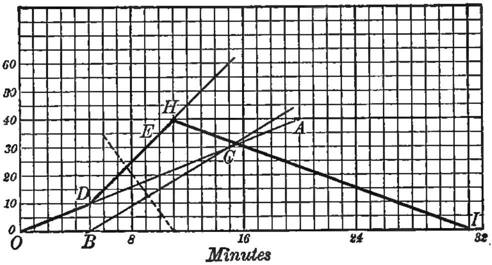


FIG. 36

We shall represent minutes on the horizontal scale and gallons of water in the tank on the vertical scale. The increase of water due to each faucet is at a constant rate, and the decrease when the outlet is opened is also at a constant rate. The amount of water in the tank due to each cause separately is, therefore, a linear function of the time, and their algebraic sum is also a linear function of the time. The first faucet begins at $t = 0$ (when w, the amount of water in tank, is 0) to supply water at a uniform rate which would supply 40 gallons in 20 minutes. The amount of water in the tank due to the first faucet alone would then be represented at any instant by the straight line OA joining the points O (0, 0) to A (20, 40). The second faucet begins when $t = 5$ to supply water at the rate of 30 gallons in 10 minutes. If this second faucet were operating alone, the water supplied by it at a given instant would be represented by the straight line joining B (5, 0) to C (15, 30). In the actual problem from the instant $t = 5$, the two faucets are running simultaneously. The sum of the two functions is then represented by the line-segment DE, where $D = (5, 10)$ and $E = (10, 20 + 15) = (10, 35)$. This line may be obtained graphically from the figure. When $t = 11$, a new factor enters, which reduces the amount of water in the tank to zero at $t = 32$. You may now finish the discussion. The required graph is the broken line $ODHI$. What would be the effect on the graph if one or both faucets were turned off at $t = 20$, the outlet remaining open?

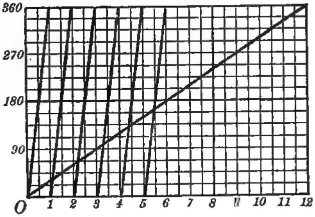
EXERCISES

1. A man on horseback rides from a place A to a place B, 15 miles distant, in 2 hours. When he is 4 miles from A, he passes a lady walking in the same direction. The man remains at B $\frac{1}{2}$ hour and then returns to A on foot. After walking 1 hour, he meets the lady on her way to B. If the man walks at the rate of 3 miles per hour, find the rate at which the lady is walking and at what time she left A.

2. A man starts at A to walk through B to a place C. At the same time a second man starts to walk from B to C. The first man reaches B in $1\frac{1}{4}$ hours, while the second man has only walked $\frac{3}{4}$ as far in this time. In how many hours will the first man overtake the second?

3. Represent graphically on the same drawing the motion of the hour and the minute hand of a clock and use the drawing to determine approximately at what time the two hands are in the same position.

[HINT: The hands are together at twelve. Lay off the hours from 12 (or, 0) to 12 on the horizontal axis and the angles in degrees that either hand makes with the 12 o'clock position on the vertical axis. Each hand moves at a constant angular speed. The graph of the hour hand is then a straight line joining the points (0, 0), (12, 360). The minute hand goes from 0 to 360 in 1 hour. The graph during the first hour is then a straight line joining (0, 0) to (1, 360). At 1 o'clock the graph begins at (1, 0) and goes to (2, 360) and so on.]



4. At what time between five and six o'clock are the hands of a watch together?

5. At what time between two and three o'clock are the hands of a watch opposite to each other? At right angles?

6. At what time between four and five o'clock are the hands of a clock at right angles? (Two solutions.)

7. A and B start to walk towards each other from two towns 15 miles apart. A walks at the rate of 3 miles per hour but rests one hour at the end of the first 6 miles. B walks 4 miles per hour but rests two hours at the end of the first 4 miles. In how many hours do the two men meet?

8. Two men can do a certain piece of work in 12 and 15 days respectively. After the first man has worked 3 days alone, the two men finish the work. How long do they work together? *Ans.* 5 days.

9. A messenger boy riding a bicycle at the rate of 9 miles per hour is sent to overtake a man on horseback riding 6 miles per hour. How long will it take the boy to overtake the man if the man had a start of 4 miles?

59. Explicit and Implicit Functions. We have hitherto considered functions which were defined explicitly by an expression involving the variable. Thus the relation between $y°$ Fahrenheit and $x°$ Centigrade was expressed by the relation

$$y = \tfrac{9}{5}x + 32.$$

Now let us consider the equation $2x - 3y + 7 = 0$. This equation also defines a functional relation between two variables. To every value of x corresponds a definite value of y, and, conversely, to every value of y corresponds a definite value of x. But, the equation does not express one of the variables explicitly as a function of the other. In fact the form of the equation gives no indication which of the variables is to be considered as the independent variable and which as the function. Such a relation is said to define a function *implicitly*.

From such an implicit relation we can derive the expression of either variable as an explicit function of the other. Thus from $2x - 3y + 7 = 0$ follows at once

$$y = \tfrac{2}{3}x + \tfrac{7}{3} \text{ and } x = \tfrac{3}{2}y - \tfrac{7}{2}.$$

The first of these equations expresses y as an explicit function of x, and the second expresses x as an explicit function of y.

60. The General Equation $Ax + By + C = 0$. Any linear relation between two variables x and y can be written in the form

$$Ax + By + C = 0. \tag{1}$$

For example, the relation just discussed in the preceding article is obtained from this general relation by placing $A = 2$, $B = -3$, $C = 7$. Equation (1) always defines y as a linear function of x, except when $B = 0$. In this case the term involving y drops out and the equation reduces to $Ax + C = 0$, and we cannot speak of y as a function of x.

But, if $B \neq 0$, we have $By = -Ax - C$, or

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which is of the form $y = mx + b$. Hence we conclude:

Any equation of the form $Ax + By + C = 0$ defines y as a linear function of x for all values of A, B, C except $B = 0$.

61. The General Equation of the Straight Line. It follows from the result of the last section, that the locus of the equation

$$Ax + By + C = 0,$$

when interpreted geometrically in rectangular coördinates, is a straight line, except perhaps when $B = 0$, when the equation takes the form $Ax + C = 0$. In this case, if $A = 0$ also, the equation reduces to $C = 0$, and it completely disappears. If A is not zero, we may solve the equation for x and obtain,

$$x = -\frac{C}{A},$$

or

$$x = \text{a constant}.$$

Now, the locus of a point whose abscissa is constant is a line parallel to the y-axis and at a distance equal to the constant from it. Thus the locus of $x = -3$ is a line parallel to the y-axis, and three units to the left of it.

The case $B = 0$ is not then an exception, and we have the following theorem.

Every equation of the form $Ax + By + C = 0$, *when represented geometrically by means of rectangular coördinates, represents a straight line. If* $B = 0$, *the line is parallel to the y-axis; if* $A = 0$, *the line is parallel to the x-axis; if* $C = 0$, *the line passes through the origin.*

Prove the last two statements of this theorem.

We may also state the following theorems.

Every straight line in the plane may be represented by an equation of the form $Ax + By + C = 0$.

The loci of $Ax + By + C = 0$ *and* $k(Ax + By + C) = 0$ $(k \neq 0)$ *are identical.*

The proofs of these theorems are left as exercises.

62. Analytic Geometry. We have thus far used the notion of coördinates to give a geometric interpretation to algebraic relations. It is possible to reverse the process and use the connection established between algebra and geometry, for the study of geometry. This method of studying geometry by algebraic means is called ***analytic geometry.*** In the following sections we proceed to develop certain analytic methods applicable to the straight line. The results are, in a large measure, merely a restatement from a different point of view of the results already obtained.

63. Straight Lines. We have already seen that the graphs of equations $Ax + By + C = 0$ and $y = mx + b$ (§ 52), when represented by means of rectangular coördinates, are straight lines. In § 60 we saw that the first of these equations could be put in the form of the second, provided $B \neq 0$. Thus when an equation of the form $Ax + By + C = 0$ *is solved for* y, *the coefficient of* x *is the slope, and the constant term is the* y*-intercept.*

The slope of the line connecting the two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is (§§ 51–53)

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We see geometrically that a line is determined when we know its slope and a point on the line. To determine the equation of this line, if (x_1, y_1) is the given point and m the given slope, we proceed as follows. Let (x, y) be any variable point on the line. Then, equating slopes, we have

$$\frac{y - y_1}{x - x_1} = m,$$

that is,

$$y - y_1 = m(x - x_1)$$

is the required equation.

It is left as an exercise to prove that the equation of the straight line through the two given points (x_1, y_1), (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

if $x_1 \neq x_2$.

64. Parallel Lines. In Fig. 37 let (1) and (2) be two parallel lines with slopes m_1 and m_2. Construct the positive segments P_1Q_1 and P_2Q_2 from the points P_1 and P_2 on lines (1) and (2) respectively, and complete the right triangles $P_1Q_1R_1$ and $P_2Q_2R_2$. We then have

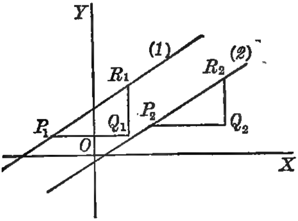


FIG. 37

$$m_1 = \frac{Q_1R_1}{P_1Q_1} \text{ and } m_2 = \frac{Q_2R_2}{P_2Q_2}.$$

If the lines are parallel, Q_1R_1 and Q_2R_2 are either both positive or both negative; m_1 and m_2 have then the same sign. They have the same magnitude since the triangles $P_1Q_1R_1$ and $P_2Q_2R_2$ are similar. Hence,

If two lines are parallel, their slopes are equal, i.e. $m_1 = m_2$. Conversely, *if the slopes of two lines are equal, the lines are parallel.*

The proof of this statement is left as an exercise.

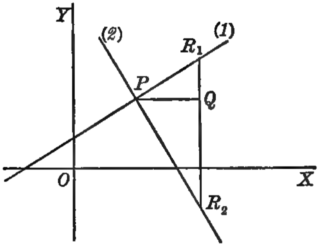


FIG. 38

65. Perpendicular Lines. In Fig. 38 let (1) and (2) be two perpendicular lines with slopes m_1 and m_2 and *let the units on the two axes be equal.* From the intersection P of the two lines construct the positive horizontal segment PQ of any convenient length. Through Q draw the vertical line R_1R_2. We then have

$$m_1 = \frac{QR_1}{PQ} \text{ and } m_2 = \frac{QR_2}{PQ}.$$

Therefore the slopes have opposite signs. Why? Also from the right triangle R_1R_2P we have $\overline{PQ}^2 = |QR_1| \cdot |QR_2|$. Therefore * $m_1m_2 = -1$ and

$$m_1 = -\frac{1}{m_2}.$$

That is, *if the units on the coördinate axes are equal, perpendicular lines have slopes which are negative reciprocals of each other.* Conversely, *if the slopes of two lines are negative reciprocals of each other, the lines are perpendicular, provided the units on the coördinate axes are equal.* The proof of this statement is left as an exercise. Why is it necessary to assume the units equal?

66. Illustrative Examples. EXAMPLE 1. Find the equation of the straight line through the point (4, 7) and having the slope -2.

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* This proof presupposes that neither m_1 nor m_2 is zero, *i.e.* the lines are not parallel to the coördinate axes, and the result obtained does not apply to such lines. However, two lines parallel to the x- and y- axes have equations of the form $y =$ a constant and $x =$ a constant, respectively, and hence can be recognized at once.

We have at once from § 63, $y - 7 = -2(x - 4)$
or

$$2x + y - 15 = 0.$$

Example 2. Find the equation of the straight line through the points $P_1(2, 4)$, $P_2(-5, 6)$.

The slope $$m = \frac{6 - 4}{-5 - 2} = -\frac{2}{7}.$$

From § 63 the equation of the line is $y - 4 = -\frac{2}{7}(x - 2)$ or $7y + 2x - 32 = 0$.

Example 3. Express the temperature measured by $y°$ Fahrenheit as a function of $x°$ Centigrade.

We know that when $y = 32$, $x = 0$: *i.e.* $P_1(0, 32)$ is a point on the graph. In the same way we have $P_2(100, 212)$ a point of the graph. Therefore the equation of the line connecting these points is

$$\frac{212 - 32}{100 - 0} = \frac{y - 32}{x - 0}$$

or

$$y = \tfrac{9}{5}x + 32 \quad \text{(See § 55, Example 1.)}$$

Example 4. Find the equation of the straight line through the point $(2, -5)$ and parallel to the line $2y + 4x - 5 = 0$.

The slope of the given line is -2 (§ 63). Therefore the equation of the required line is $y + 5 = -2(x - 2)$ or $2x + y + 1 = 0$.

Example 5. Find the equation of the straight line through the point $(1, -2)$ and perpendicular to the line $3x - y + 2 = 0$.

The slope of the given line is 3. Therefore the slope of the required line is $-\frac{1}{3}$ (§ 65). The equation of the required line is $y + 2 = -\frac{1}{3}(x - 1)$ or $x + 3y + 5 = 0$.

EXERCISES

1. What is the meaning of the constants m and b in the equation $y = mx + b$?

2. What is the effect on the line $y = mx + b$ if b is changed while m remains fixed? If m changes when b remains fixed?

3. Describe the effect on the line $y - y_1 = m(x - x_1)$ if m changes while x_1, y_1 remain fixed: also describe the effect if x_1, y_1, vary while m remains fixed.

4. What is the equation of the line

(*a*) whose slope is 3 and whose y-intercept is 2; *Ans.* $y = 3x + 2$.
(*b*) whose slope is 4 and whose y-intercept is -3;
(*c*) whose slope is 0 and whose y-intercept is -1;
(*d*) whose slope is 0 and whose y-intercept is 0?

5. Describe the positions of lines (*c*) and (*d*) in Ex. 4.

6. Define "y-intercept of a line." What is meant by the "x-intercept"?

7. For each of the following lines give x-intercept, y-intercept, and slope:

(*a*) $2x - 3y = 7$. *Ans.* $\frac{7}{2}$; $-\frac{7}{3}$; $\frac{2}{3}$. (*c*) $2x - y + 5 = 0$.
(*b*) $x + y - 2 = 0$. (*d*) $4x + y = 0$.

8. Is a straight line determined if we know its intercepts? Try each of the equations $2x - y = 4$ and $2x - y = 0$.

9. Find the equation of the line joining the two points (2, 1) and (−3, 1); of the line joining the points (4, 2) and (4, −3).

10. Which of the following lines are parallel?

(*a*) $2x - y - 4 = 0$. (*c*) $4x - 2y - 1 = 0$.
(*b*) $y + 2x + 3 = 0$. (*d*) $2y + 4x + 5 = 0$.

11. Are the points (1, 5), (−1, 1), (2, 6) on the line $y = 2x + 3$?

12. What is the equation of the line which is parallel to $y = 2x + 3$ and passes through the origin? perpendicular to $y = 2x + 3$ and passes through the origin?

13. Determine k so that

(*a*) the line $2x + 3y + k = 0$ shall pass through the point (0, 1);
Ans. -3.
(*b*) the line $2x + 3y + k = 0$ shall have a y-intercept equal to 2;
(*c*) the line $2x + 3y + k = 0$ shall have an x-intercept equal to 5.

14. Find the equations of the sides of the triangle whose vertices are $(3, 4)$, $(-1, 2)$, $(-4, -5)$.

Ans. $x - 2y + 5 = 0$; $9x - 7y + 1 = 0$; $7x - 3y + 13 = 0$.

15. Find the equations of the sides of the quadrilateral whose vertices are $(-2, 1)$, $(3, -1)$, $(-2, 4)$, $(1, 7)$.

16. What intercepts does the line through the points $(2, -7)$ and $(4, -5)$ make on the axes?

17. Find the equation of the line which passes through the point $(4, -2)$ and whose slope is 6.

18. A line has the slope 2 and passes through the point $(-1, 2)$. What are its intercepts?

19. What is the equation of the line which passes through $(-5, 5)$ if its y-intercept is -3? *Ans.* $8x + 5y + 15 = 0$.

20. Write the equations of the lines which make the following intercepts on the x- and y-axes.

(a) 2 and -4; (b) -7 and -3; (c) 4 and 5; (d) 0 and 0.

21. If the x- and y-intercepts of a line are a and b, prove that the equation of the line can be written in the form

$$\frac{x}{a} + \frac{y}{b} = 1, \quad (ab \neq 0).$$

[This equation is called the ***intercept form*** of the equation of a straight line.]

22. Solve Ex. 20 by using the result of Ex. 21. Does the formula hold in Ex. 20, (d)? Explain.

23. Find the equation of the straight line through the point $(4, -5)$ parallel to the line $2x - y + 7 = 0$; through the same point, perpendicular to the line $2x - y + 4 = 0$. *Ans.* $y = 2x - 13$; $2y = -x - 6$.

24. Prove that the lines $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel. State this theorem in words.

25. Prove that the lines $Ax + By + C = 0$ and $Bx - Ay + D = 0$ are perpendicular. State this theorem in words.

26. Prove that the lines $Ax + By + C = 0$ and $Mx + Ny + P = 0$ are perpendicular if and only if $AM + BN = 0$.

27. Show that the points $(-8, 0)$, $(-4, -4)$, $(-4, 4)$, and $(4, -4)$ are the vertices of a trapezoid.

28. The Réaumur thermometer is graduated so that water freezes at 0° and boils at 80°. Find the equation of the line that represents the reading R of the Réaumur thermometer as a function of the corresponding reading C of the Centigrade thermometer.

29. A printer asks 75 cents to set the type for a notice and 3 cents per copy for printing. The total cost is what function of the number of copies printed? Draw the graph of this function.

30. Express the value of a \$1000 note at 6% simple interest as a function of the time in years. Is this a linear function?

31. A cistern is supplied by a pipe that supplies water at the rate of 30 gallons per hour. Assuming that the amount A of water in the cistern is connected with the time t by a linear relation, find this relation if $A = 1000$ when $t = 10$. What is A when $t = 0$?

32. In stretching a wire it is assumed that the elongation e is connected with the tension t by means of a linear relation. Find this relation if $t = 20$ lb. when $e = 0.1$ in. and $t = 60$ lb. when $e = 0.3$ in.

67. Systems of Straight Lines. An equation of the first degree in x and y, and containing an arbitrary constant, represents in general an infinite number of straight lines. For the equation will represent a straight line for each value of the constant. All the lines represented by an equation of the first degree containing an arbitrary constant are said to form a ***system of lines.*** The arbitrary constant is called the ***parameter*** of the system. Thus the equation $y = -3x + b$ represents the system of straight lines with slope -3. (See Fig. 39.) The equation $y - 2 = m(x - 1)$ represents the system of straight lines through the point (1, 2).* (See Fig. 40.)

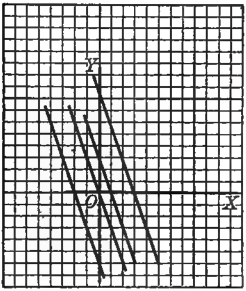

FIG. 39

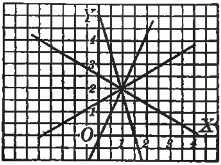

FIG. 40

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* It represents every line of this system except the one parallel to the y-axis. Why?

68. Pencil of Lines. All the lines in a plane which pass through a given point are said to form a ***pencil of lines.*** The point is called the ***center*** of the pencil. If $Ax + By + C = 0$, and $A'x + B'y + C' = 0$ are any two lines of the pencil, then

$$(3) \qquad (Ax + By + C) + k(A'x + B'y + C') = 0,$$

where k is an arbitrary constant, represents a line of the pencil. This is true because the equation (3)

(a) is of the first degree in x and y and therefore represents a straight line;

(b) is satisfied by the coördinates of the point of intersection of the two given lines. Why?

Example 1. Find the equation of the line through the point $(2, -5)$ and parallel to $4x + 2y + 5 = 0$.

The system of lines parallel to $4x + 2y + 5 = 0$ is given by the equation $y = -2x + k$. Since we want the particular line of the system that passes through the point $(2, -5)$, the equation must be satisfied by these coördinates. It follows that, $-5 = -4 + k$ or $k = -1$.

Therefore, $y = -2x - 1$ is the desired equation.

Example 2. Find the equation of the line through the point $(4, -1)$ and perpendicular to $3x + 2y - 5 = 0$.

The system of lines perpendicular to $3x + 2y - 5 = 0$ is given by the equation $y = \frac{2}{3}x + k$. Since we want the line of the system that passes through the point $(4, -1)$, we have $k = -\frac{11}{3}$. Therefore, the desired equation is

$$y = \tfrac{2}{3}x - \tfrac{11}{3} \quad \text{or} \quad 2x - 3y - 11 = 0.$$

Example 3. Find the equation of the line through the intersection of $2x + y - 4 = 0$ and $x + y - 1 = 0$, and perpendicular to $x + 2y = 3$.

Any other line through the intersection of the given lines is

$$(4) \qquad (2x + y - 4) + k(x + y - 1) = 0$$

or

$$x(2+k)+y(1+k)+(-4-k)=0.$$

The slope of this line is $-(2+k)/(1+k)$ and this must be equal to the negative reciprocal of the slope of the straight line $x+2y=3$. Therefore,

$$-\frac{2+k}{1+k}=2 \quad \text{and} \quad k=-\frac{4}{3}.$$

Substituting this value in equation (4) and simplifying, we have $2x-y-8=0$, the required equation.

EXERCISES

1. Find the equation of the straight line through the point (1, 5) and parallel to $2x+3y-9=0$; perpendicular to $2x+3y-9=0$.

Ans. $2x+3y-17=0$; $3x-2y+7=0$.

2. Find the equations of the altitudes of the triangle whose vertices are (2, 8), (4, − 5), (3, − 2).

3. Find the equation of the straight line through the intersection of $10x+5y+11=0$ and $x+2y+14=0$ which is perpendicular to $x+7y+1=0$; parallel to $3x-7y=1$.

4. Find the equation of the straight line through the intersection of $x+2y-4=0$ and $x-3y+1=0$ which is perpendicular to $3x-2y+4=0$; parallel to $x-y=0$.

5. Find the equation of the straight line through the intersection of $x+y-1=0$, $x-3y+4=0$ and

(*a*) through the point (1, 1); *Ans.* $x+5y-6=0$.
(*b*) parallel to the line $x+2y-9=0$;
(*c*) perpendicular to the line $4x-5y=0$;
(*d*) through the intersection of $3x+4y-8=0$ and $x-5y+7=0$.

6. Find the equation of the straight line which passes through the point

(*a*) (0, 0) and is parallel to $2x-y+4=0$;
(*b*) (1, 2) and is perpendicular to $3x-2y-1=0$;
(*c*) (− 1, 2) and is parallel to $x-y-4=0$.

7. Find the equation of the line which passes through the intersection of $x-y+2=0$ and $x+y=0$ and through the intersection of $x+y+2=0$, $x-y=0$.

8. Find the equation of the straight line through the intersection of $x - 2y + 7 = 0$ and $2x - y + 3 = 0$ and

(*a*) parallel to the x-axis;
(*b*) parallel to the y-axis.

9. Find the equation of the straight line which passes through the intersection of $3x - y + 2 = 0$ and $x + y = 5$ and which

(*a*) passes through the origin;
(*b*) is parallel to $x - 4y + 3 = 0$;
(*c*) is perpendicular to $3x - 2y + 4 = 0$.

69. Intersection of Two Lines. Simultaneous Equations.

We have just seen that linear equations in one or two variables are represented in rectangular coördinates by straight lines. We now wish to determine the coördinates of the point of intersection of two lines whose equations are given. That is, algebraically, we wish to find a set of values for x and y which satisfy both equations.

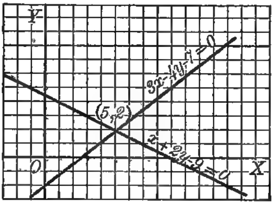

FIG. 41

EXAMPLE 1. Solve the equations

(5) $3x - 4y = 7.$
(6) $x + 2y = 9.$

Multiplying equation (6) by 2 and adding the result to equation (5), we obtain $5x = 25$, or $x = 5$.

Likewise multiplying equation (6) by 3 and subtracting the result from equation (5), we have $-10y = -20$, or $y = 2$. The set of values $x = 5$, $y = 2$ is seen to satisfy both equations and is called the *solution* of the given equations. If we plot lines (5) and (6) (Fig. 41), we see from their graph that the coördinates of their point of intersection are (5, 2).

Therefore, a method of solving two linear equations in one or two variables is to plot the lines represented by each equation, and then determine from the graph the coördinates of the point of intersection. The algebraic method of first eliminating one variable and then the other has the advantage over the geometric method in that it is always accurate. Instead of eliminating twice, the value found for either variable can be substituted in either equation, and the value of the second variable determined.

Example 2. Solve the equations

$$(7) \qquad x - 2y = 3.$$

$$(8) \qquad x - 2y = -5.$$

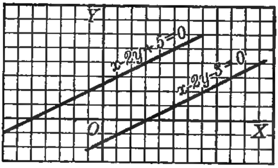

Fig. 42

Subtracting the second equation from the first, we obtain $0 = 8$. That is, there are no values of x and y satisfying both equations. Such equations are said to be *inconsistent* or *incompatible.* We see that lines (7) and (8) have the same slope, but different y-intercepts, and therefore are parallel lines.

Example 3. Solve the equations

$$(9) \qquad x - y = 2.$$

$$(10) \qquad 2x - 2y = 4.$$

Multiplying the first equation by 2 and subtracting the second from it, we have $0 = 0$. If equation (10) be divided by 2, equations (9) and (10) are seen to represent the same relation between x and y, and are not therefore sufficient to determine x and y. We can assign to either variable an arbitrary value and then find the corresponding value for the other variable. The equations can, therefore, be said to have an infinite number of solutions. Such equations are called *dependent.* The graphs of these equations are coincident lines.

Let us now consider the general equations

$$(11) \qquad a_1x + b_1y = c_1,$$

$$(12) \qquad a_2x + b_2y = c_2,$$

where none of the constants are zero. Eliminating y, we obtain $(a_1b_2 - a_2b_1)x = c_1b_2 - c_2b_1$. Eliminating x, we obtain $(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$. Now if $a_1b_2 - a_2b_1 \neq 0$, we have

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

If, however, $a_1b_2 - a_2b_1 = 0$, *i.e.* $a_2/a_1 = b_2b_1$, we cannot solve for x and y. Denoting the common value of these quotients by k, we have $a_2 = ka_1$, $b_2 = kb_1$. Then equations (11) and (12) become $a_1x + b_1y = c_1$, and $ka_1x + kb_1y = c_2$.

We must now distinguish two cases according as $c_2 = kc_1$ or $c_2 \neq kc_1$. In the former case, by dividing out k, we see that the equations are dependent and have an infinite number of solutions. In the latter case they are inconsistent, and thus are not satisfied simultaneously by any values of x and y.

Discuss the cases that arise if some of the constants are zero.

EXERCISES

Find, when they exist, the coördinates of the points of intersection of the following lines. Check your answer from a graph.

1. $4x + 2y = 9.$
$2x - 5y = 0.$

2. $3x + 4y = 12.$
$x - y = 5.$

3. $x + 2y = 3.$
$2x + 4y = 6.$

4. $x - 2y = 7.$
$2x - 4y = 5.$

5. $x + 4y = 1.$
$2x + 8y = 2.$

6. $x - 2y = 7.$
$-x + 2y = 3.$

In the following exercises are the lines concurrent? If so, what point have they in common?

7. $x + 2y = 3.$
$x - y = 0.$
$5x - y = 4.$

8. $x - y = -1.$
$2x + y = 3.$
$3x - 2y = 1.$

9. $x + 2y = 5.$
$5x - y = 3.$
$2x + y = 4.$

10. $x - 2y = 3.$
$5x - y = 2.$
$2x + 3y = 1.$

In the following exercises, find k so that the lines shall be concurrent.

11. $x + y = 2.$
$2x - y = 1.$
$4x + y = k.$ *Ans.* 5.

12. $2x - y = 0.$
$x + 3y = 7.$
$x + ky = 5.$

13. $3x - y = 4.$
$x + y = 0.$
$5x - 2y = k.$

14. The sides of a triangle have for their equations $2x + y = 5$, $x - y = 10$, $-2x + y = 6$. What are the coördinates of the vertices of this triangle? What are the equations of the altitudes?

15. Find the equation of the straight line through (2, 1), (–1, 2), using the equation $Ax + By + C = 0$. [HINT: Solve for A/C and B/C.]

16. Find the equation of the straight line through (4, 7) and having the slope 3, using the equation $Ax + By + C = 0$.

17. It has been shown experimentally, that the length l of a wire in feet under a tension of p pounds, is $l = a + bp$, where a and b are constants. Find a and b if $l = 190$ when $p = 270$, and that $l = 190.2$, when $p = 450$.

18. The readings T and S of two gas meters are connected by the equation $T = a + bS$. Determine a and b when we know that $S = 10$, when $T = 300$, and $S = 100$, when $T = 420$.

19. The pull in pounds to lift a load l in pounds with a pulley is given by the relation $p = al + b$, where a and b are constants. Find a and b when it is known that a pull of 8 pounds lifts a load of 40 pounds, while it takes a pull of 2 pounds to hold the rope on when no weight is attached.

70. Equations Containing More than Two Unknowns.

It is easy to see that the methods employed in § 69 for solving a system of two simultaneous equations, each containing two unknown quantities, may also be employed for solving a system of three or more equations, involving as many unknown quantities as there are independent equations.

EXAMPLE. Solve the equations

$$7x + 3y - 2z = 16. \tag{13}$$

$$5x - y + 5z = 31. \tag{14}$$

$$2x + 5y + 3z = 39. \tag{15}$$

Adding three times (14) to (13) gives

$$22x + 13z = 109. \tag{16}$$

Adding five times (14) to (15) gives

$$27x + 28z = 194. \tag{17}$$

Solving equations (16) and (17) by the methods of § 69, we have $x = 2$, $z = 5$. Substituting these values in (13), we obtain $y = 4$. It is readily seen that $x = 2$, $y = 4$, $z = 5$ satisfies equations (13), (14), (15).

The cases in which three simultaneous equations in three unknowns have no solution, or an infinite number of solutions, will be discussed in Chapter XXI.

EXERCISES

Solve the following simultaneous equations :

1. $2x + 4y + z = 12.$
$3x + y - z = 3.$
$x + y + z = 7.$

2. $x + y + z = 13.$
$x - 2y + 4z = 10.$
$3x + y - 3z = 5.$

3. $2x - 3y - z = 2.$
$5x + 2y + z = -8$
$x - 2y - z = 2.$

4. $x + 8y - 4z = 9.$
$3x + 3y - z = 6.$
$5x + 2y - 2z = 7.$

5. $w + x + y = 15.$
$x + y + z = 18.$
$w + y + z = 17.$
$w + x + z = 16.$

6. $x + y = 4.$
$2x + z = 4.$
$y - z = 3.$

7. If A and B can do a piece of work in 10 days, and A and C in 8 days, and B and C in 12 days, how long will it take each to do the work alone?

8. Three towns A, B, and C are situated at the vertices of a triangle. The distance from A to B via C is 76 miles; from A to C via B 79 miles; from B to C via A 81 miles. Find the distance from A to B, from B to C, from C to A.

9. In a triangular track meet the following was the final score:

SCORE	FIRST PLACE	SECOND PLACE	THIRD PLACE	TOTAL
College A . . .	5	3	3	37
College B . . .	2	4	1	23
College C . . .	2	2	6	22

How many points did each place count?

10. Two passengers traveling from town A to town B have 500 pounds of baggage. The first pays $ 1.75 for excess above weight allowed, the second $ 1.25. If the baggage belonged to the last passenger, he would have to pay $ 4 excess. How much baggage is allowed to a single passenger?

11. A crew can row 4 miles downstream and back again in $1\frac{1}{2}$ hours, or 6 miles downstream and half way back in the same time. What is the rate of rowing in still water, and what is the rate of the current?
Ans. 6 miles per hour; 2 miles per hour.

12. Two trains are scheduled to leave two towns A and B, m miles apart, at the same time, and to meet in h hours. The train leaving A was k hours late in starting, so the trains met n hours later than the scheduled time. What is the rate at which each train runs?

13. Two men are running at uniform rates on a circular track 150 feet in circumference. When they run in opposite directions, they meet every 5 seconds. When they run in the same direction, they are abreast every 25 seconds. What are their rates?

14. Find a, b, c, so that $y = a + bx + cx^2$ shall be satisfied by $(2, 1)$, $(1, 0)$, $(3, -6)$.

15. Find a, b, c, so that $\dfrac{6x^2 - x - 3}{x^3 - x} = \dfrac{a}{x-1} + \dfrac{b}{x+1} + \dfrac{c}{x}$.

CHAPTER IV

THE QUADRATIC FUNCTION

I. GRAPHS OF QUADRATIC FUNCTIONS

71. The General Quadratic Polynomial $ax^2 + bx + c$. Having considered in some detail the linear function $mx + b$ and its geometric interpretation, we now turn our attention to a similar study of the quadratic function, *i.e.* a function expressed by a polynomial of the second degree in one variable. Such polynomials are, for example, $x^2 + 1$, $100 + 50\,t - 16.1\,t^2$, etc. The *general form* of such a polynomial is $ax^2 + bx^2 + c$, where a, b, c are constants and $a \neq 0$. Such functions abound in practice. Thus, if a projectile be shot vertically upward from the top of a tower 100 ft. high, with an initial velocity of 50 ft. per second, the distance s (in feet) from the ground at the end of t seconds, is given approximately by the polynomial

$$s = 100 + 50\,t - 16.1\,t^2.$$

The general formula for the distance s from the ground at the end of t seconds of a projectile shot vertically upward is (approximately)

$$s = s_0 + v_0 t - \tfrac{1}{2}\,gt^2,$$

where s_0 is the distance from the ground when $t = 0$, v_0 is the initial velocity, and g is the so-called "gravitational constant," which varies slightly from place to place but is approximately equal to 32.2 when the distance s is measured in feet and the time is measured in seconds.

72. The Function x^2. We consider first the simplest of all quadratic functions, viz. the function $y = x^2$. A brief tabular representation of this function is as follows:

x	0	1	2	3	4	-1	-2	-3	-4
y	0	1	4	9	16	1	4	9	16

If we plot these points, we obtain Fig. 43, in which we notice that the points seem to be arranged according to some regular law. We may insert additional points by calculating values of y for values of x between those already used. Thus for $x = 1.5$, $y = 2.25$ and $x = -1.5$ $y = 2.25$. These points are also marked on the figure. In general we see that for $x = a$ and also for $x = -a$, we have $y = a^2$. Geometrically this means that the graph is symmetrical with respect to the y-axis, *i.e.* if the part of the graph on the right of the y-axis is turned about the y-axis until it falls in the original plane, it will coincide with the part on the left of the y-axis. Moreover, since x^2 is positive (or zero) for all real values of x, no part of the graph will be below the x-axis.

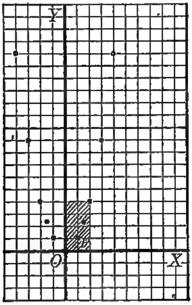

FIG. 43

Keeping these facts in mind we shall make a more detailed study of this function and its graph, by considering values of x which are closer together. We shall confine ourselves to values of x between $x = 0$ and $x = 2$. The corresponding values of y, for all values in this range at intervals of 0.1 of a unit, are given in the following table:

x	y	x	y	x	y	x	y
0.1	0.01	0.6	0.36	1.1	1.21	1.6	2.56
0.2	0.04	0.7	0.49	1.2	1.44	1.7	2.89
0.3	0.09	0.8	0.64	1.3	1.69	1.8	3.24
0.4	0.16	0.9	0.81	1.4	1.96	1.9	3.61
0.5	0.25	1.0	1.00	1.5	2.25	2.0	4.00

We cannot, with any accuracy, insert in Fig. 43 the corresponding points of the graph. We therefore adopt a procedure analogous to the use of a magnifying glass, in order to separate the points. This we have done in Fig. 44 by choosing the unit on each axis 10 times as large as in Fig. 43. We then see that there is no difficulty in plotting all the points given in the above table.

Let us study more carefully the immediate neighborhood of some point on the graph, for example, $P(1, 1)$. We shall magnify the shaded area in Fig. 44 in the ratio 10 : 1 and make use of the following table:

x	y	x	y	x	y	x	y	x	y
0.90	.8100	0.95	.9025	1.00	1.0000	1.05	1.1025	1.10	1.2100
0.91	.8281	0.96	.9216	1.01	1.0201	1.06	1.1236		
0.92	.8464	0.97	.9409	1.02	1.0404	1.07	1.1449		
0.93	.8649	0.98	.9604	1.03	1.0609	1.08	1.1664		
0.94	.8836	0.99	.9801	1.04	1.0816	1.09	1.1881		

It will be noted that the points of the graph now lie almost on a straight line (Fig. 45). We have drawn a straight line through P for the purpose of comparison. If we should desire a more detailed representation in the neighborhood of the point P, we should calculate the values of y for values of x between $x = .99$ and $x = 1.01$ and draw anew a small portion of the figure about P under a tenfold increase of the unit. We would then find that the points would hardly be distinguishable from the points on a straight line.

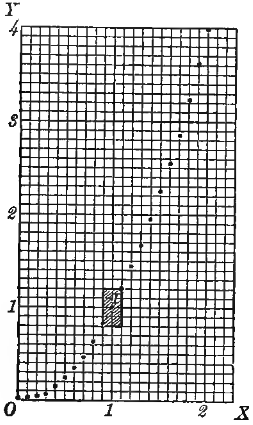

FIG. 44

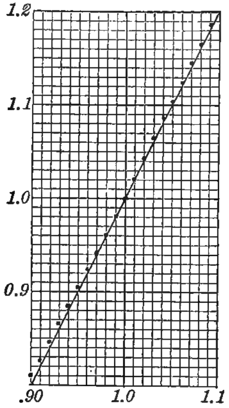

FIG. 45

Similar conclusions might be reached near any other point on the graph. It is of course impossible to prove this for each separate point by separate calculations. To prove the fact generally we proceed as follows. Let x_1 be *any* particular value of the variable x and y_1 the corresponding value of the function y; then $y_1 = x_1^2$. Now suppose that the value x is increased or decreased by a certain amount, which we shall call Δx (a decrease means that Δx is negative). The new value of x is then $x_1 + \Delta x$ and the corresponding value of the function is $(x_1 + \Delta x)^2$. This new value of the function differs from the original value of the function, y_1, by a certain amount which we shall call Δy. We then have

$$y_1 + \Delta y = (x_1 + \Delta x)^2$$

$$= x_1^2 + 2\,x_1\Delta x + \Delta x^2;$$

but

$$y_1 = x_1^2.$$

Therefore, by subtraction,

$$\Delta y = 2\,x_1\Delta x + \Delta x^2$$

or

$$\Delta y = (2\,x_1 + \Delta x)\,\Delta x. \tag{1}$$

Since formula (1) is true for every value of x_1, it follows that Δy approaches zero when Δx approaches zero. This means that in the neighborhood of the point (x_1, y_1) we can find new points on the graph whose x and y differ from those of the given point by as little as we please. This simply means that the set of all points of the graph of $y = x^2$ form a set of points with no gaps between them; they form what we may call a continuous line or curve.*

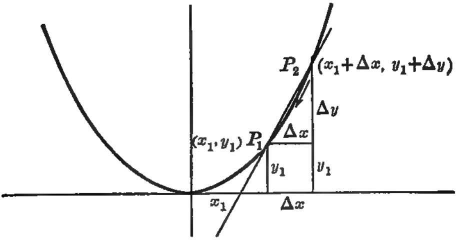

FIG. 46

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* A function is said to be *continuous* for a value $x = x_1$, if when Δx approaches 0 the corresponding Δy also approaches 0. See footnote on p. 19.

Further, equation (1) gives the relation,

$$\frac{\Delta y}{\Delta x} = 2\,x_1 + \Delta x \qquad (\text{if } \Delta x \neq 0).$$

From the graph (Fig. 46) we clearly see that this change ratio is the slope of the line joining the points $P_1(x_1, y_1)$ and $P_2(x_1 + \Delta x, y_1 + \Delta y)$.* If the latter point approaches the former along the curve, *i.e.* if we let Δx become numerically smaller and smaller, then the change ratio $\Delta y / \Delta x$ will differ less and less from $2\,x_1$. Indeed, we may choose Δx sufficiently small (without making it zero) so that $\Delta y / \Delta x$ will differ from $2\,x_1$ by less than any previously assigned amount.

Geometrically this means that in the immediate neighborhood of the point P_1 on the graph of $y = x^2$, the points of the graph lie very near to the straight line through P_1 whose slope is $2\,x_1$. From a somewhat different point of view, we can let the secant joining the points $P_1\,(x_1, y_1)$ and $P_2\,(x_1 + \Delta x, y_1 + \Delta y)$ on the graph rotate about P_1 in such a way that Δx, and therefore Δy, become smaller and smaller and the secant approaches a definite position through P_1 whose direction has the slope $2\,x_1$. This line is by definition *tangent* to the graph at P_1, or the graph is tangent to the line at P_1; the point P_1 is called the *point of contact.* Combining the above results we have:

The graph of the function $y = x^2$ is a continuous curve, above the x-axis, symmetrical with respect to the y-axis, and passing through the origin. At any point $P_1\,(x_1, y_1)$ on the curve, the straight line with slope $2\,x_1$ passing through this point is tangent to the curve.

73. Further Observations regarding the Function $y = x^2$. The preceding result tells us that when $x = 1$, the slope of the tangent is 2. Reference to Fig. 45 will verify this result for the straight line there drawn, since this line has the slope 2. In Fig. 47 we have reproduced Fig. 43 except that we have replaced the several points plotted in the earlier figure by a continuous curve and have drawn the tangent at the point

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* This follows also directly from the formula $m = (y_2 - y_1)/(x_2 - x_1)$.

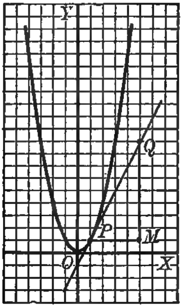


FIG. 47

$P(1, 1)$. Knowing that the slope of the tangent is 2, we can easily construct the tangent. Starting from P we lay off any convenient distance PM to the right and then lay off double this distance MQ upward. The line PQ is then the required tangent. A similar process leads to the construction of the tangent at any other point of the curve.

From the fact that the slope of the tangent at any point on the curve whose abscissa is x_1 is $2x_1$, we see that as x_1 increases numerically the slope increases numerically, that is, the curve becomes steeper and steeper the farther we go from the origin. Also the slope is positive when x_1 is positive and negative when x_1 is negative. This means that going from left to right the curve slopes downward at the left of the origin, and upward at the right of the origin. When $x = 0$, the slope is zero, that is to say, the tangent is parallel to the x-axis (here it coincides with the x-axis).

Hitherto in our drawings we have chosen the unit on the y-axis to be equal to that on the x-axis. This renders it impossible to draw the graph of the function $y = x^2$ for large values of x, without making it of unwieldy size. However nothing prevents us from choosing the unit on the y-axis smaller than that on the x-axis, and in Fig. 48 we have chosen it one tenth as large. A tabular representation is as follows:

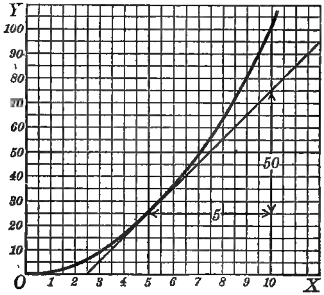


FIG. 48

x	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10
y	1	4	9	16	25	36	49	64	81	100

In this case the slopes of the tangents are, respectively,

$$\pm 2,\ \pm 4,\ \pm 6,\ \pm 8,\ \pm 10,\ \pm 12,\ \pm 14,\ \pm 16,\ \pm 18,\ \pm 20.$$

We have drawn the tangent at the point for which $x = 5$, and have drawn the graph only for positive values of x.

EXAMPLE. Find the equation of the tangent to the graph of $y = x^2$ at the point (3, 9).

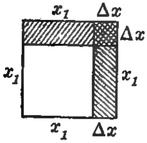
The slope of the tangent at the point (x_1, y_1) is $2\,x_1$. Therefore at (3, 9) the slope is 6. The equation of the tangent is, therefore, $y - 9 = 6\,(x - 3)$ or $y = 6\,x - 9$.

EXERCISES

1. Discuss the functions $y = -x^2$; $y = 2x^2$; $y = -2x^2$.

2. Construct for the point (2, 4) of the function $y = x^2$ a figure analogous to Fig. 45. (Use a table of squares.)

3. Use the adjoining figure to give a geometric interpretation of the equation $\Delta y = 2x_1\Delta x + \Delta x^2$. The function $y = x^2$ is here interpreted as the area of the square whose side is x.



4. If in the function $y = x^2$ we take $x = 3$, how small must Δx be taken in order that Δy shall be numerically less than 0.01? if we take $x = 15$? Is the difference between these two results to be expected in view of the nature of the graph?

5. Draw the tangents to the curve $y = x^2$ at the points for which $x = 0$, $\pm \frac{1}{2}$, ± 1, $\pm \frac{3}{2}$, ± 2.

6. If x is the radius of a circle and y is its area, prove that the change ratio $\Delta y / \Delta x$ approaches the length of the circle as Δx approaches zero.

7. Find the equations of the tangents to the curve $y = x^2$ at the following points: (1, 1); (2, 4); (− 1, 1); (− 2, 4). Construct the tangents at these points.

8. The line perpendicular to the tangent at the point of contact is called the ***normal*** to the curve at this point. Find the equations of the normals to $y = x^2$ at the points (1, 1); (2, 4); (− 1, 1); (− 2, 4). Construct each normal making use of its slope.

Ans. For the point (1, 1): $x + 2y - 3 = 0$.*

9. Find the slope of the tangent to $y = 3x^2$ at the point whose abscissa is x_1. What is the value of this slope at the point (1, 3)?

10. Find the equations of the tangent and the normal (see Ex. 8) to $y = 3x^2$ at the points (3, 27); (− 2, 12).

11. Find the points where the slope of the curve $y = x^2$ has the values − 1; 2; 10.

12. 1 cu. ft. of water weighs 66.4 lb. What must be the diameter x of a cylindrical can such that 1 in. of water contained in it will weigh y oz.? Plot the graph and find x when $y = 50$. Find y when $x = 8$.

* Assuming the units on the axis to be equal.

74. The General Quadratic Function $y = ax^2 + bx + c$. We may now dispose of the general case. Let

$$y = ax^2 + bx + c \ (a \neq 0)$$

be any quadratic function (in the case $y = x^2$, a was 1, while b and c were 0). Let x increase from the value x_1 to the value $x_1 + \Delta x$, and suppose that this change in the value of x changes the value of the function from y_1 to $y_1 + \Delta y$. We desire to calculate the value of Δy and of the change ratio $\Delta y/\Delta x$. We have

$$y_1 + \Delta y = a(x_1 + \Delta x)^2 + b(x_1 + \Delta x) + c,$$

and

$$y_1 = ax_1^2 + bx_1 + c.$$

Subtracting, we obtain

$$(2) \qquad \Delta y = (2\,ax_1 + b + a\Delta x)\,\Delta x,$$

and

$$(3) \qquad \frac{\Delta y}{\Delta x} = 2\,ax_1 + b + a\Delta x \quad (\text{if } \Delta x \neq 0).$$

Equation (2) shows that Δy can be made numerically as small as we please, by choosing Δx *near enough to* 0. Hence we may say:

Every function of the form $y = ax^2 + bx + c$ is continuous.

Equation (3) shows that the change ratio $\Delta y/\Delta x$ approaches as a limit the value $2ax_1 + b$ as Δx approaches 0. Hence we may say:

The slope of the tangent *to the curve $y = ax^2 + bx + c$ at the point whose abscissa is x_1 is equal to* $\boldsymbol{2ax_1 + b}$.

75. General Properties of the Function $ax^2 + bx + c$. The discussion in the preceding section and the exercises have furnished us with some information regarding some special functions of the form $ax^2 + bx + c$.

It will now be shown that whenever the term in x^2 is positive (*i.e.* a is positive) the graph of the function is an inverted

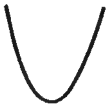


FIG. 49

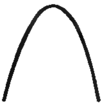


FIG. 50

arch as in Fig. 49 and that whenever the term in x^2 is negative (*i.e.* a is negative) the graph is an arch like the one in Fig. 50.

To prove this we need only consider the slope of the tangent to the curve as the point of contact moves along the curve. We have just seen that the slope of the tangent is given by the formula $m = 2ax_1 + b$ at the point whose abscissa is x_1. There is just one point on the curve for which this slope is zero, viz. the point whose abscissa is

$$x_1 = -\frac{b}{2a} \qquad (a \neq 0).$$

Now let us write the slope m of the tangent in the form

$$m = 2a\left(x_1 + \frac{b}{2a}\right).$$

The number in the parenthesis, *i.e.*, $x_1 + b/(2a)$, is positive when $x_1 > -b/(2a)$ and negative when $x_1 < -b/(2a)$. Geometrically this means that this parenthesis represents a positive number for points to the right of the straight line $x = -b/(2a)$ and a negative number for points to the left of this straight line.

CASE 1: $a > 0$. If a is positive, the slope m is positive for points to the right of the line $x = -b/(2a)$ and negative for points to the left of this line.

In other words, for all points of the graph to the left of the line $x = -b/(2a)$ the tangent slopes downward (as we go from left to right) and for all points to the right of this line the tangent slopes upward. The point for which $x = -b/(2a)$ has its tangent parallel to the x-axis. This point is called the ***minimum point*** of the graph (Fig. 51).

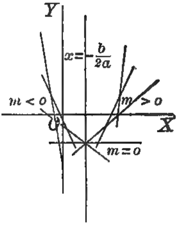


FIG. 51

CASE 2: $a < 0$. Suppose on the other hand that a is negative. The slope m is then negative when $x_1 + b/(2a)$ is positive and positive when $x_1 + b/(2a)$ is negative. The slope is therefore positive when $x_1 < -b/(2a)$ and negative when $x_1 > -b/(2a)$. At the single point for which $x = -(b/2a)$ the tangent is parallel to the x-axis. This point is called the ***maximum point*** of the graph (Fig. 52).

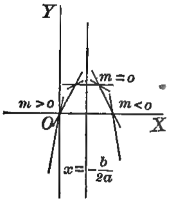


FIG. 52

When $x = -b/(2a)$ the function $y = ax^2 + bx + c$ has a minimum value if $a > 0$ and a maximum value if $a < 0$.

The curve represented by the function $y = ax^2 + bx + c$ is symmetrical with respect to the line $x = -b/(2a)$.

The proof is left as an exercise.

HINT. Show that the points which have abscissas $-b/(2a) + h$ and $-b/(2a) - h$ have the same ordinate.

76. Definitions. The curve represented by an equation of the form

$$y = ax^2 + bx + c$$

is called a ***parabola***. The lowest (or highest) point on this curve, *i.e.* the point for which $x = -b/(2a)$, is called the ***vertex***. The straight line through the vertex and perpendicular to the tangent at the vertex is called the ***axis*** of the curve. The parabola is symmetrical with respect to its axis.

77. To draw the Graph of a Parabola $y = ax^2 + bx + c$. The preceding discussion enables us to draw the graph of a quadratic function without plotting many points.

EXAMPLE 1. Sketch the graph of $y = 2x^2 - 6x + 5$.

The slope of the tangent at (x_1, y_1) is, by § 74, $m = 4x_1 - 6$. The vertex of the curve is the point for which $4x_1 - 6 = 0$, *i.e.* the point for which $x_1 = 3/2$; the corresponding value of y is $1/2$ and the vertex is therefore the point $(3/2, 1/2)$. This point is the minimum point of the curve. We plot this vertex V, draw the horizontal tangent at this point and the vertical axis. We desire a few more points and their tangents on each side of the axis and then we can draw the curve. For example, we have

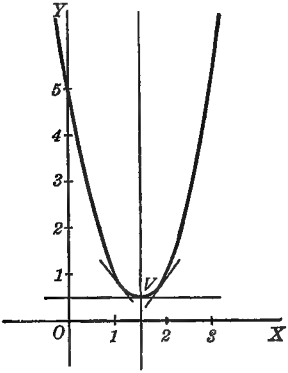


FIG. 53

x	y	m
1	1	-2
2	1	2
0	5	-6

EXAMPLE 2. Sketch the graph of $y = -x^2 + 4x + 5$.

The slope of the tangent at (x_1, y_1) is $m = -2x_1 + 4$. The vertex of the curve is at the point for which $-2x_1 + 4 = 0$, *i.e.* for which $x_1 = 2$. The corresponding value of y_1 is 9. Therefore the vertex, which is the maximum point of the graph, is at (2, 9). The graph is given in Fig. 54.

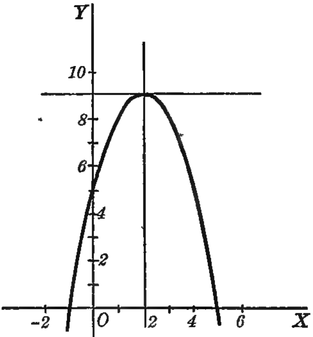


FIG. 54

EXERCISES

1. Tell which of the following functions have a maximum and which have a minimum value. Find this value in each case and the corresponding value of x.

(*a*) $2x^2 + 8x - 9$.

Ans. Minimum value: -17, when $x = -2$.

(*b*) $3x^2 + 8x - 6$.

(*c*) $-5x^2 + 10x - 12$.

(*d*) $3x^2 + 6x - 7$.

(*e*) $-x^2 + 1$.

2. Find the coördinates of the vertex and the equation of the axis of each of the following parabolas. Sketch the curves.

(*a*) $y = 2x^2 + 5x + 3$.

(*b*) $y = 3x^2 + 9x - 6$.

(*c*) $y = -5x^2 + 10x - 12$.

Ans. $V = (1, -7)$; axis, $x = 1$.

(*d*) $y = 3x^2 + 6x - 7$.

(*e*) $y = -x^2 + 1$.

3. The area of a certain rectangle in terms of the length of its side x is $A = x(100 - 2x)$. Find x so that this area shall be a maximum.

4. A point moves on a straight line so that its distance s from a fixed point O on the line at any time t is given by one of the equations below. Draw the (s, t) graph and in each case show that the variable point reaches, on one side of O, a maximum absolute distance from O. Find this maximum distance. Does this maximum absolute distance correspond to a maximum or a minimum value of s?

(*a*) $s = t^2 - 4t + 3$.

(*b*) $s = 2t^2 - 8t + 10$.

(*c*) $s = 3 + 6t - 4t^2$.

5. Find the equations of the tangent and the normal* to the curve $y = x^2 - 3x + 1$ at the point $(1, -1)$. *Ans.* $y = -x$; $y = x - 2$.

6. Find the equations of the tangent and the normal* to the curve $y = -2x^2 + 3x - 1$ at the point $(1, 0)$.

7. Find the equations of the tangent and the normal* to the curve $y = -2x^2 + 4x - 1$ at the maximum point. *Ans.* $y = 1$; $x = 1$.

8. Find the equations of the tangent and the normal* to the curve $y = 3x^2 - 6x + 1$ at its vertex.

---

* See Ex. 8, p. 108.

78. The Graph of $y - k = a(x - h)^2$. The fact that the graphs of functions of the form $y = ax^2 + bx + c$, all have the same general shape but are differently located with respect to the coördinate axes suggests that many of these graphs may consist of curves, which might be brought into coincidence by a suitable motion in the plane. That this is indeed the case results from the following considerations, which lead to a general principle of far-reaching importance.

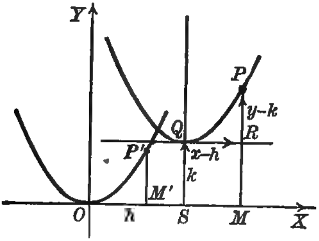


FIG. 55

Suppose the graph of the equation $y = ax^2$ is moved parallel to itself through a distance and direction which carries the point O to the point Q (h, k). What will be the equation between the x and y of any point P on the curve in its new position, the axes of coördinates remaining in their original positon? This question is readily answered. Let P' be the position of P before it was moved. The equation $y = ax^2$ then tells us that $M'P' = a \cdot \overline{OM'}^2$ for every position of P' on the curve in its old position. After the motion, the directed segments OM' and $M'P'$ become respectively the directed segments QR and RP. Hence, for every point P on the curve in its new position we have

$$(4) \qquad RP = a \cdot \overline{QR}^2.$$

If the coördinates of P are (x, y) we have $x = OM$, $y = MP$ and

$$QR = x - h, \qquad RP = y - k.$$

Therefore, by (4), the curve in its new position is the graph of the equation

$$(5) \qquad y - k = a \cdot (x - h)^2.$$

While we have applied these considerations to the function

$y = ax^2$, the reasoning is general; consequently we may formulate the following principle:

General Principle. *If in any equation between x and y we replace x by $x - h$ and y by $y - k$, the graph of the new equation is obtained from the graph of the original equation by moving the latter graph parallel to itself in such a way that the point O moves to the point (h, k).*

We shall have occasion to apply this principle often in the future.

79. Transformation by Completing the Square. At present we may use the principle just stated to prove that *the parabolas $y = ax^2 + bx + c$ and $y = ax^2$ are congruent curves.*

This follows at once from the preceding general principle, if we prove that the equation

$$y = ax^2 + bx + c \tag{6}$$

can be written in the form

$$y - k = a(x - h)^2. \tag{7}$$

To do this we write (6) as follows:

$$y = a\left(x^2 + \frac{bx}{a} + \qquad\right) + c,$$

and then *complete the square* on the terms in the parentheses by adding the term $b^2 / (4a^2)$. In order to leave the value of y unchanged we must also subtract $a \times b^2 / (4a^2) = b^2 / (4a)$ from the expression. This gives

$$y = a\left(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}, \tag{7'}$$

or

$$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2.$$

This is of the form (7) for the values

$$h = -\frac{b}{2a}, \qquad k = -\frac{b^2 - 4ac}{4a}.$$

The operation just performed is called the ***transformation by completing the square.*** It is found serviceable in a variety of situations. It may be used to advantage in connection with numerical examples.

EXAMPLE. *Discuss the graph of* $y = -2x^2 + 8x - 9$.
We first write

$$y = -2(x^2 - 4x + \quad) - 9;$$

and then

$$y = -2(x^2 - 4x + 4) - 9 + 8$$

or

$$y + 1 = -2(x - 2)^2.$$

The graph is then obtained from the graph of $y = -2x^2$ by moving the latter parallel to itself so that its vertex moves to the point (2, −1).

EXERCISES

1. By reducing to the form $y - k = a(x - h)^2$, discuss the graphs of each of the following functions.

(*a*) $y = 2x^2 + 12x + 2$.
(*b*) $y = 4x^2 + 6x - 9$.
(*c*) $y = -3x^2 + 9x + 10$.
(*d*) $y = 2x^2 - 7x + 3$.
(*e*) $y = -4x^2 + 7x + 2$.
(*f*) $y = -3x^2 - 8x + 10$.

2. Show that the equation of the straight line $y - y_1 = m(x - x_1)$ may be derived from the equation $y = mx$ by the general principle of § 78.

3. The results of § 79 furnish a proof of the fact previously derived, that the vertex of the parabola $y = ax^2 + bx + c$ is at the point for which $x = -b/(2a)$. Explain.

4. Equation (7′) proves that if $a > 0$, the value $x = -b/(2a)$ gives the minimum value to y; also that if $a < 0$, the value $x = -b/(2a)$ gives the maximum value to y. Explain without using the graph.

Write the following equations in the form $a(x - h)^2 + b(y - k)^2 = c$, where a, b, c, h, and k are constants.

5. $x^2 - 4x + 2y^2 - 8y = 2$.
Ans. $(x - 2)^2 + 2(y - 2)^2 = 14$.
6. $-2x^2 + 4x + y^2 - 4y - 3 = 0$.
7. $4x^2 - 4x + 2y^2 - 3y + 1 = 0$.
8. $x^2 + y^2 - 4y = 2$.
9. $x^2 - 8x + y^2 = 0$.
10. $3x^2 - 4x - y^2 + 2 = 0$.

II. APPLICATIONS OF QUADRATIC FUNCTIONS

80. Maxima and Minima. We have seen that a quadratic function $ax^2 + bx + c$ has either a maximum or a minimum value according as a is negative or positive. Numerous applications involve the problem of finding this maximum or minimum value and the corresponding value of x, as the following examples show.

EXAMPLE 1. A rectangular piece of land is to be fenced in and a straight wall already built is available for one side of the rectangle. What should be the dimensions of the rectangle in order that a given amount of fencing will inclose the greatest area?

Before beginning the solution proper we should note carefully the significance of the problem. The length of the fence being given, we may use it to inclose rectangles of a variety of shapes, as indicated by the dotted lines in Fig. 56. Some rectangle whose shape is between those indicated will inclose the maximum area. To determine this shape is our problem. To do this, it is necessary to express the area (the quantity we wish a maximum) as a function of one variable.

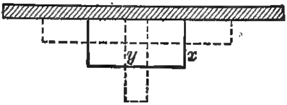


FIG. 56

SOLUTION: Let the dimensions of the rectangle be x and y and suppose the given length of fencing is L. We then have

$$2x + y = L. \tag{8}$$

The area inclosed is $A = xy$, which from (8) becomes

$$A = x(L - 2x) = Lx - 2x^2.$$

Plotting this function, we have the parabola in Fig. 57. We desire to find the value of x corresponding to the vertex V of this parabola, for this gives the greatest value to A. The slope m of the tangent is given by the equation $m = L - 4x$, and this is zero (tangent horizontal at V) when $x = \frac{1}{4}L$. For this value of x, $y = \frac{1}{2}L$. The maximum area is therefore obtained when the width is one half of the length. The maximum area is $\frac{1}{8}L^2$ square units.

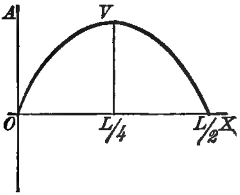


FIG. 57

Example 2. Three streets intersect so as to inclose a triangular lot ABC. The frontage of the lot on BC is 180 ft. and the point A is 90 ft. back of BC. A rectangular building is to be constructed on this lot so as to face BC. What are the dimensions of the ground plan which will give the maximum floor area?

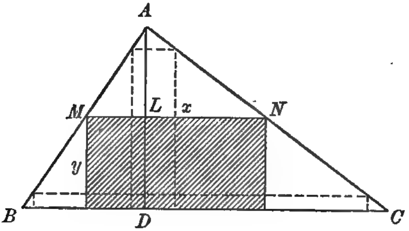


Fig. 58

In Fig. 58 we have drawn the lot ABC and have indicated by dotted lines two extreme plans. The ground plan sought must be somewhere between these two extremes. To determine its dimensions we proceed as follows:

Let x and y be the length of the sides of the ground plan. The floor area (neglecting the thickness of the walls) is

$$A = xy. \tag{9}$$

In order to express A, for which we seek a maximum, in terms of x alone, we now proceed to express y in terms of x. The triangles ABC and AMN are similar. (Why?) Hence we have

$$\frac{MN}{BC} = \frac{LA}{DA}. \qquad \text{(Why?)}$$

This gives

$$\frac{x}{180} = \frac{90 - y}{90},$$

whence

$$y = -\tfrac{1}{2}x + 90. \tag{10}$$

From (9) and (10) we obtain

$$A = 90\,x - \tfrac{1}{2}x^2.$$

This expresses the floor area as a function of the side x. The slope of the tangent to the graph is given by

$$m = 90 - x$$

and this slope is zero when $x = 90$, which in turn gives (by (10)) $y = 45$, and therefore $A = 4050$. The maximum area is then 4050 sq. ft. and this is obtained by making the building 90 ft. long and 45 ft. deep.

Draw the graph of the function $A = 90\,x - \frac{1}{2}x^2$.

We may note that in both of these examples, the function of which the maximum was sought was obtained as a function of two variables. The conditions of the problem, however, made it possible to express one of these variables in terms of the other and thus to obtain the desired function as a *quadratic function of one variable,* whereupon the solution was readily effected. The difficulty in this type of problem is usually in connection with the elimination of all but one of the variables. To solve such a problem it is necessary to keep in mind the following steps.

(1) Decide, and express *in words,* of what function a maximum or a minimum value is to be found.

(2) Express this function algebraically.

(3) If this expression contains more than one variable, use the conditions of the problem to find a relation or relations connecting these variables.

(4) By means of the relation or relations found, eliminate all but one of the variables from the function of which a maximum or minimum value is sought.

(5) Proceed with the algebraic computation.

EXERCISES

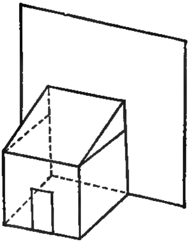
1. The number 100 is separated into two parts such that the product of the parts is a maximum. Find the parts and the corresponding product. *Ans.* 50, 50, 2500.

Is it possible to separate 100 into two parts such that the product of the corresponding parts is a minimum? Explain.

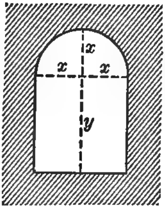
2. Prove that the rectangle of given perimeter which has the maximum area is a square.

3. Find the greatest rectangular area that can be inclosed by 100 yd. of fence.

4. Separate 20 into two parts such that the sum of their squares will be a minimum.



5. A man desires to build a shed against the back of his house, the ground plan to be a rectangle. The roof is to be 1 ft. higher in the back than in the front (see the adjoined figure). He has on hand enough siding to cover 253 sq. ft. Allowing 18 sq. ft. for a door and assuming that the height from the ground to the lowest part of the roof is 8 ft., what should be the dimensions of the ground plan in order to get the greatest floor area?



6. An underground conduit is to be built, the cross section of which is to have the shape of a rectangle surmounted by a semicircle. If the cost of the masonry is proportional to the perimeter, and if the perimeter is 30 ft., what should be the dimensions of the cross section in order that the conduit will have a maximum capacity?

7. The same problem as in Ex. 6 with the perimeter of the cross section given as a ft.

8. Determine the greatest rectangle that can be inscribed in a given acute angled triangle whose base is $2b$ and whose altitude is $2a$.

*9. In the corner of a field bounded by two perpendicular roads a spring is situated 8 chains from one road and 6 chains from the other. How should a straight path be run by this spring and across the corner so as to cut off as little of the field as possible?

Ans. 12 and 16 chains from the corner.

81. Table of Squares. We have stated that the more important functions have been tabulated (§ 28). The function x^2 is one of these. Tables of squares are very helpful in shortening computation. A comparatively rapid method of constructing such a table is given in Ex. 2 below. Here we may make use of our knowledge of the function x^2 to see that for a sufficiently small interval in such a table, we are justified in using linear interpolation (§ 56). Indeed we have seen that in a sufficiently small neighborhood of any point on the graph of $y = x^2$, the graph differs as little as we please from a straight line. (See Fig. 45.) For example, if in the second table on p. 100 we confine ourselves to only three-place accuracy, we find that the successive differences in the function are almost proportional to the corresponding differences in the variable. We give in the adjoined table an extract from the table mentioned. From this table we may conclude that

---

* The function whose minimum is sought is not in this case quadratic An approximate solution may be obtained graphically. The solution may be computed by finding the slope of the graph from the definition of slope.

x	x^2
.94	.884
.95	.903
.96	.922
.97	.941
.98	.960
.99	.980

$$(.953)^2 = .909.$$

This result is accurate only to the third decimal place.

EXERCISES

1. Find by interpolation from the above table the following:

$$(.954)^2;\ (.981)^2;\ (9.66)^2;\ (9.89)^2.$$

2. Compute by actual multiplication the squares of all the integers from 31 to 40. This method of computing a table of squares becomes very laborious. Write the results obtained from your computation in a column, and write opposite each pair of successive squares their difference as shown in the adjoined beginning of such a table. These differences are called the *first differences of the table*. Do you observe any regularity in the formation of these differences? Prove in general the law here suggested.

x	x^2	Difference
31	961	63
32	1024	65.
33	1089	
34		
35		
36		

[HINT. Consider the difference between k^2 and $(k+1)^2$.]
Use this law to construct a table of squares from 41 to 100.

3. If the successive differences of the *first differences* are formed, we obtain the so-called *second differences.* Prove that in a table of squares of successive integers the successive *second differences* are all equal to 2. The first differences, therefore, have the character of a linear function. Hence show how to compute the exact value of $(32.6)^2$ from the value of $(32)^2$ and $(33)^2$. This process is known as *quadratic interpolation.*

III. QUADRATIC EQUATIONS

82. Definitions. An equation of the form $ax^2 + bx + c = 0$, where a, b, and c are constants and $a \neq 0$, is called a **quadratic equation.**

A value of x which when substituted in the equation $ax^2 + bx + c = 0$ makes both members identical is called a ***root.***

Example 1. Is 3 a root of the equation $2x^2 - 5x + 6 = 0$?

Substituting 3 for x, we find $2.3^2 - 5.3 + 6 = 9$ and not 0. Therefore 3 is not a root.

Example 2. Determine k so that one root of $2kx^2 - 3x + 5 = 0$ shall be 1.

Since 1 is to be a root, we have $2k - 3 + 5 = 0$, or $k = -1$. The equation then becomes $-2x^2 - 3x + 5 = 0$.

83. The Roots of $ax^2 + bx + c = 0$. It follows from § 79 that the equation $ax^2 + bx + c = 0$ may be written in the form

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a},$$

provided $a \neq 0$. Dividing by a and solving for $(x + b/(2a))$, we have

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2},$$

or

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a};$$

hence

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{11}$$

We have shown up to this point that if $ax^2 + bx + c$ has the value 0, then x must have one of the values given in equation (11). We need still to prove the converse: *If*

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad or \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

then $ax^2 + bx + c$ *will have the value* 0. This can be done by substituting the values of x in turn in the expression ax^2+bx+c and simplifying the resulting expressions.*

The last part of this proof is *essential.* We know that the converse of a true theorem may be false.† The first part of our discussion proved that *no other* values of x than those given by (11) will satisfy the equation $ax^2 + bx + c = 0$, but it did not prove that either of these values *does satisfy* the given equation.

Equations (11) may be used as a formula for solving a quadratic equation. Thus, solving

$$2x^2 - 5x - 13 = 0$$

where $a = 2$, $b = -5$, $c = -13$, we have

$$x = \frac{5 \pm \sqrt{25 - 4(2)(-13)}}{4},$$

or

$$x = \frac{5 \pm \sqrt{129}}{4}.$$

Solution by Factoring. If the factors of a quadratic equation may be found readily, one may proceed as in the following example.

Example. Solve $x^2 - 3x + 2 = 0$.

This equation may be written in the form

$$(x-2)(x-1) = 0.$$

Therefore,

$$x - 2 = 0 \quad \text{or} \quad x - 1 = 0,$$

i.e.

$$x = 2 \quad \text{or} \quad x = 1.$$

Why? See § 48.

---

* The converse can be proved at present only if $b^2 - 4ac$ is not negative. Why ?

† Thus the converse of the true statement, "A horse is an animal," would be the false statement, "An animal is a horse."

EXERCISES

Determine whether the roots of the following equations are as stated.

1. $x^2 - 5x + 6 = 0$; 2, 3.

2. $x^2 + 5x - 6 = 0$; 1, 2.

3. $x^2 - 12x + 30 = 0$; 5, 6.

4. $2x^2 - 5x + 3 = 0$; 1, -2.

5. $x^2 - 7 = 0$; $\sqrt{7}$, $-\sqrt{7}$.

6. $7x^2 - 2x + 51 = 0$; 0, 1.

In the following equations determine k so that the number beside the equation is a root. Find the other root.

7. $x^2 + 2kx - 5 = 0$; 1.

Ans. $k = 2$; other root $= -5$.

8. $kx^2 - 5x + k^2 - 1 = 0$; 0.

9. $kx^2 - 5kx + 11 = k$; 2.

Solve the following equations by means of the formula and also by completing the square:

10. $(ax + b)^2 = 6x$.

11. $(x - 5)(7x - 3) = 12$.

12. $\frac{y+5}{7} - \frac{y^2-5}{3} = 6$.

13. $x^2 + kx - dx^2 + h = 0$.

14. $m^2x^2 + m(n - p)x - mp = 0$.

15. $sx^2 + tx - p = 0$.

16. $\frac{x^2}{4} - \frac{(3x+2)^2}{1} = 1$.

17. $3(5x^2 - 10) + 2x - 5 = 0$.

18. $x^2 + (p - q)x - pq = 0$.

Solve the following equations by factoring:

19. $x^2 - 8x + 15 = 0$.

20. $x^2 - 14x + 48 = 0$.

21. $12 - x - x^2 = 0$.

22. $3x^2 - 17x + 10 = 0$.

23. $5x + 14 = x^2$.

24. $abx^2 + a^2x + b^2x + ab = 0$.

25. A cross-country squad ran 6 miles at a certain constant rate and then returned at a rate 5 miles less per hour. They were 50 minutes longer in returning than in going. At what rate did they run?

Ans. 9 miles per hour.

26. When a single row of rivets is used to join together two boiler plates, the distance p between the centers of the rivets is given by the formula

$$p = 0.56\frac{d^2}{t} + d,$$

where t is the thickness of the plate and d is the diameter of a rivet hole in inches. In a certain make of boiler the rivets are 1 inch apart and the plate is $\frac{1}{2}$ inch thick. Find the diameter of the rivet holes.

27. How high is a box that is 6 ft. long, 2 ft. wide, and has a diagonal 8 ft. in length?

28. The effective area E of a chimney is given by the formula $E = A - 0.6\sqrt{A}$, where A is the measured area. Find the measured area when the effective area is 25 square feet.

29. Two men can row 12 miles downstream and back again in 5 hours. If the current is flowing at the rate of 1 mile per hour, how fast can the men row in still water?

30. Find the outer radius of a hollow spherical shell an inch thick whose volume is $76\pi/3$ cubic inches.

[HINT. The volume of a sphere is $4\pi r^3/3$.]

84. Graphic Solution. EXAMPLE. Solve $x^2 - 4x + 3 = 0$ graphically.

Let us plot the graphs of $y = x^2$, $y = 4x - 3$ with reference to the same set of axes (Fig. 59). We see that the two graphs intersect in two points, the coördinates of which satisfy both equations. Therefore the abscissas of these points are values of x which make the right-hand members equal, *i.e.*, for which

$$x^2 = 4x - 3$$

or

$$x^2 - 4x + 3 = 0.$$

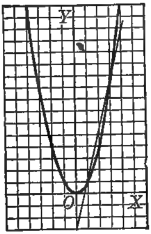


FIG. 59

The roots are seen to be 1 and 3.

If the line and the parabola were tangent, what would you say concerning the roots? If the line and parabola do not meet, what would you say concerning the roots?

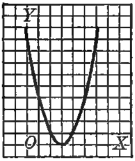


FIG. 60

This problem may be solved graphically in an entirely different way. We will plot the curve $y = x^2 - 4x + 3$ (Fig. 60). The abscissas of the points where this graph meets the x-axis are roots of the original equation. Why?

Describe the roots if the parabola touches the x-axis. What would you say concerning the roots if the parabola did not meet the x-axis?

85. General Theorems. 1. *If r is a root of the equation $ax^2 + bx + c = 0$, then $x - r$ is a factor of $ax^2 + bx + c$.*

Dividing $ax^2 + bx + c$ by $x - r$, we obtain:

$$\begin{array}{r}
\underline{x - r\,\big|\,ax^2 + bx + c\,\big|\,ax + (b + ar)} \\
\underline{ax^2 - arx \qquad\qquad\qquad\quad} \\
(b + ar)x + c \qquad\quad \\
\underline{(b + ar)x - (b + ar)r} \\
c + br + ar^2.
\end{array}$$

Therefore

$$ax^2 + bx + c = [ax + (b + ar)][x - r] + c + br + ar^2.$$

But, by hypothesis, r is a root; therefore, $ar^2 + br + c = 0$; hence

$$ax^2 + bx + c = [ax + (b + ar)][x - r].$$

2. Prove that *if $x - r$ is a factor of $ax^2 + bx + c$, then $x = r$ is a root of $ax^2 + bx + c = 0$.*

3. Prove that *if the expression $ax^2 + bx + c$ is divided by $x - r$, the remainder is $ar^2 + br + c$.*

THE DISCRIMINANT OF THE QUADRATIC. In § 83 we saw that the roots of the equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The expression under the radical, namely, $\boldsymbol{b^2 - 4ac}$, is called the ***discriminant*** of the equation, because it enables us to discriminate as to the nature of the roots. From geometric considerations we know that a quadratic equation with *real coefficients* a, b, c may have either two real distinct roots, two real equal roots, or no real roots at all. The above formula enables us to see the same truth algebraically.

If $\boldsymbol{b^2 - 4ac = 0}$, we say that there are two *real* and *equal* roots, each being $-b/2a$.

If $\boldsymbol{b^2 - 4ac > 0}$, there are two *real* and *unequal* roots.

If $\boldsymbol{b^2 - 4ac < 0}$, there are no real roots. The roots of such an equation are called imaginary or complex. The properties of such numbers will be discussed fully in Chap. XVIII.

If the discriminant $b^2 - 4ac$ is a perfect square and the coefficients a, b, c are rational numbers, then the roots are rational.

By finding the value of the discriminant we may determine the nature of the roots of the quadratic without solving the equation. Thus, in the equation $3x^2 + 4x - 3 = 0$, the discriminant is 52 and we conclude that the roots are real, unequal and irrational.

Relation of Roots to Coefficients. Let the roots of the equation $ax^2 + bx + c = 0$ be denoted by r_1 and r_2. That is, let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

By addition we have

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}.$$

By multiplication we have

$$r_1 r_2 = \frac{[(-b) - \sqrt{b^2 - 4ac}][(-b) + \sqrt{b^2 - 4ac}]}{4a^2}$$

$$= \frac{b^2 - b^2 + 4ac}{4a^2} = \frac{c}{a}.$$

Therefore, if we write the quadratic equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

the above results may be expressed as follows:

In a quadratic equation in which the coefficient of the x^2 term is unity, (i) the sum of the roots is equal to the coefficient of x with the sign changed; (ii) the product of the roots is equal to the constant term.

EXERCISES

Solve graphically (two ways) each of the following equations:

1. $2x^2 + 5x - 3 = 0.$ **3.** $12 - x = x^2.$ **5.** $4 - x^2 = 0.$

2. $x^2 - 8x + 15 = 0.$ **4.** $2x^2 - 3x - 5 = 0.$ **6.** $4 + x^2 = 0.$

Form the equations with the following roots:

7. $4, -5.$ *Ans.* $x^2 + x - 20 = 0.$ **9.** $2 + \sqrt{5}, 2 - \sqrt{5}.$

8. $\sqrt{7}, -\sqrt{7}.$ **10.** $c + 3b, c - 3b.$

11. What is the remainder when $3x^2 - 2x + 5 = 0$ is divided by $x - 3$? by $x + 2$? by $x - 1$? by $-x + 1$? [Hint: Use 3, § 85.]

Determine, the nature of the roots of the following equations:

12. $7x^2 - 5x = 6.$ **14.** $2y^2 + 3y + 24 = 0.$

13. $2x = 7 - 3x^2.$ **15.** $9x^2 = 4x - 5.$

Determine k so that the following equations shall have equal roots. [Hint: Place $b^2 - 4ac$ equal to zero.]

16. $kx^2 - 6x + 3 = 0.$ *Ans.* $k = 3.$ **18.** $x^2 + 2(1 + k)x + k^2 = 0.$

17. $3x^2 - 4kx + 2 = 0.$ **19.** $2kx^2 + (5k + 2)x + 4k + 1 = 0.$

20. Determine the limits on k so that equations 16–19 shall have their roots real and unequal; imaginary and unequal.

21. If x is real, show that $\dfrac{x}{x^2 - 5x + 9}$ must lie between $-\dfrac{1}{11}$ and 1.

22. A party of students hired a coach for $12, but three of the students failed to contribute towards the expense, whereupon each of the others had to pay 20 cents more. How many students were in the party?

23. Cox's formula for the flow of water in a long horizontal pipe connected with the bottom of a reservoir is

$$\frac{Hd}{L} = \frac{4v^2 + 5v - 2}{1200},$$

where H is the depth of the water in the reservoir in feet, d the diameter of the pipe in inches, L the length of the pipe in feet, and v the velocity of the water in feet per second. If a reservoir contains 49 ft. of water, find the velocity of the water in a 5-inch pipe that is 1000 ft. long.

24. It takes two pipes 24 minutes to fill a certain reservoir. The larger pipe can fill it in 20 minutes less time than the smaller. How long does it take each pipe to fill the reservoir? *Ans.* 60 min.; 40 min.

25. Prove algebraically and geometrically that if $b^2 - 4ac < 0$, the value of the function $ax^2 + bx + c$ is positive for all (real) values of x, if $a > 0$; and negative for all (real) values of x, if $a < 0$.

86. Equations involving Radicals. The method of solving problems involving radicals will be illustrated by some examples.

Example 1. Solve $\sqrt{x+2} - 2(x-1) = 0$.

Transposing the second term to the right-hand member gives

$$\sqrt{x+2} = 2(x-1).$$

Squaring,

$$x + 2 = 4x^2 - 8x + 4, \quad \text{or} \quad 4x^2 - 9x + 2 = 0.$$

Whence

$$x = 2, \text{ or } \tfrac{1}{4}.$$

Do both these values satisfy the equation?

We have shown that, if $\sqrt{x+2} - 2(x-1) = 0$, then $x = 2$ or $\frac{1}{4}$. But we cannot conclude conversely, that if $x = 2$ or $\frac{1}{4}$, then $\sqrt{x+2} - 2(x-1) = 0$.

In fact, if we substitute the values of x found in the original equation, we find that $x = 2$ is a root; but $x = \frac{1}{4}$ is not.

Example 2. Solve the equation $\sqrt{x+8} + \sqrt{x+3} = 5\sqrt{x}$.

Squaring both sides, we find

$$x + 8 + 2\sqrt{x^2 + 11x + 24} + x + 3 = 25x,$$

or

$$2\sqrt{x^2 + 11x + 24} = 23x - 11;$$

whence squaring, collecting terms, dividing by 25, we have

$$21x^2 - 22x + 1 = 0;$$

therefore, $x = 1$ or $\frac{1}{21}$.

What are the roots?

EXERCISES

Solve each of the following equations:

1. $\sqrt{x-2} - 3 = 0$.

2. $\sqrt{x-2} + 3 = 0$. *Ans.* No roots.

3. $\sqrt{x+2} - \sqrt{x+7} = -1$. *Ans.* 2.

4. $-\sqrt{4x-3} - \sqrt{x+1} = 1$.

5. $\sqrt{x+5} + \sqrt{x+10} = \sqrt{2x+15}$.

6. $\sqrt{x+b} + \sqrt{x+a} = \sqrt{2x+a+b}$.

7. $\sqrt{2x+6} - \sqrt{x+4} = \sqrt{x-4}$.

8. $\sqrt{x+3} - \sqrt{4x+1} = \sqrt{2-3x}$.

MISCELLANEOUS EXERCISES

Determine the condition existing among a, b, c so that the equation $ax^2 + bx + c = 0$ shall have:

1. One root double the other. *Ans.* $2b^2 = 9ac$.
2. The roots reciprocals of each other. *Ans.* $a = c$.
3. One root three times the other.
4. One root n times the other.
5. One root zero. *Ans.* $c = 0$.
6. One root equal to 1; 2; 3; n.
7. The roots numerically equal but opposite in sign. *Ans.* $b = 0$.

8. Find the area of the largest rectangle that can be inscribed in a triangle whose base is 20 inches and whose altitude is 15 inches, if one side of the rectangle is along the base of the triangle.

9. Separate twenty into two parts such that the product of half of one part by a quarter of the other shall be a maximum.

10. Solve the equation $y^4 - 8y^2 + 15 = 0$. [HINT: Let $y^2 = x$.]

11. Solve the equation $\left[x + \frac{1}{x}\right]^2 + \left[x + \frac{1}{x}\right] - 12 = 0$.

12. Solve the equation $x^2 + 8x + 3\sqrt{x^2 + 8x + 2} = 8$.

13. Solve the equation $\frac{x^2}{x+1} - \frac{x+1}{x^2} = \frac{7}{12}$.

14. Find k so that the roots of $(k + 2)x^2 - 2kx + 1 = 0$ are equal.

15. Without solving, determine the sum and product of the roots of each of the following equations:

(*a*) $2x^2 - 7x - 3 = 0$.
(*b*) $x^2 - 4x + 2 = 0$.
(*c*) $4x^2 - 3x + 1 = 0$.
(*d*) $2x^2 + 3x + 4 = 0$.

16. Determine k so that the sum of the roots of the equation $2x^2 + (k - 1)x + (3k - 7) = 0$ is 4. *Ans.* $k = -7$.

17. Determine k so that the product of the roots of the equation $(2k - 1)x^2 + (k + 3)x + (k^2 - 2k + 1) = 0$ is 2.

CHAPTER V

THE CUBIC FUNCTION. THE FUNCTION x^n

87. The General Cubic Function $ax^3 + bx^2 + cx + d$. Having discussed in the last chapter the general quadratic function $ax^2 + bx + c$, we now turn our attention to the general algebraic function of the third degree, *i.e.* the general cubic function. It is of the form

$$ax^3 + bx^2 + cx + d, \qquad a \neq 0.$$

88. The Function x^3. We begin with the consideration of the function $y = x^3$.

A brief tabular representation of this function is given below.

We note that the values of x^3 for negative values of x are the same in absolute value as those for the corresponding positive values, but negative. If the corresponding points are plotted with respect to a pair of rectangular axes, we obtain Fig. 61.

x	x^3
0	0.00
.5	0.12
1.0	1.00
1.5	3.36
2.0	8.00
2.5	15.62
3.0	27.00

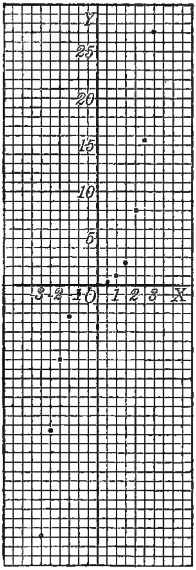


FIG. 61

The change Δy in y due to a change Δx in x is calculated as follows, where x_1 and y_1 are any pair of corresponding values of x and y:

$$(1) \qquad y_1 + \Delta y = x_1^3 + 3\,x_1^2 \cdot \Delta x + 3\,x_1 \Delta x^2 + \Delta x^3.$$

Since

$$y_1 = x_1^3,$$

this gives

$$(2) \qquad \Delta y = (3\,x_1^2 + 3\,x_1\Delta x + \Delta x^2)\,\Delta x.$$

We can now conclude that as Δx approaches zero, Δy also approaches zero; *i.e.* the function is continuous for all values of x. From (2) we obtain

$$(3) \qquad \frac{\Delta y}{\Delta x} = 3\,x_1^2 + 3\,x_1\Delta x + \Delta x^2 \quad (\text{if } \Delta x \neq 0).$$

As Δx (and, therefore, also Δy) approaches 0, this change ratio approaches $3\,x_1^2$. The slope m of the graph at the point (x_1, y_1) is, therefore,

$$(4) \qquad m = 3x_1^2.$$

This slope is positive for all values of x_1 except $x_1 = 0$. Why? The function is therefore an increasing function for all values of x except $x = 0$, *i.e.* at the origin, where the graph of the function is tangent to the x-axis. The graph is exhibited in Fig. 62, where we have drawn at certain points the tangents to the graph by means of (4) in order to insure greater accuracy.

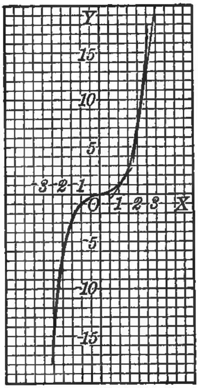


FIG. 62

89. The Functions ax^3 and $a(x-h)^3+k$. From the results of the last article and the general principles previously established, we conclude that the graph of the function

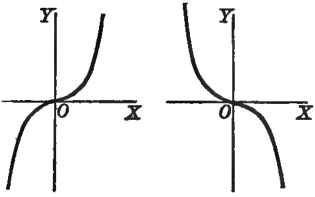
$$y = ax^3$$

is obtained from that of $y = x^3$ by stretching or contracting all the ordinates in the ratio $|a| : 1$, according as $|a|$ is greater than 1 or less than 1, and in case a is negative reversing the signs of all the ordinates (Fig. 63).* Explain the reason for this result.

The function $y = a(x - h)^3 + k$ may be written in the form

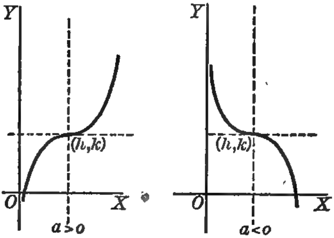
$$y - k = a(x - h)^3.$$

Its graph is accordingly (§ 78) obtained from that of $y = ax^3$ by sliding the latter graph through a distance and in a direction represented by the motion from (0, 0) to $C = (h, k)$. Explain the reason for this (Fig. 64).



$y = ax^3$

FIG. 63



$y = a(x-h)^3 + k$

FIG. 64

The slope of $y = ax^3$ at the point (x_1, y_1) is $3\,ax_1^2$. The slope of $a(x - h)^3 + k$ at the point (x_1, y_1) is $3\,a(x_1 - h)^2$. The proof of these statements is left as an exercise.

EXERCISES

1. From the graph of the function $y = x^3$, determine the volume of a cube whose edge is 0.5 in.; 0.5 ft.; 3 ft.; 1.5 cm.

2. Find the equation of the tangent and the normal to the curve $y = x^3$ at the point (2, 8); (– 1, – 1); (0, 0); (– 2, – 8).

3. Draw each of the curves $y = -x^3$, $y = 4\,x^3$, $y = 3(x - 1)^3$, $y = 2(x + 1)^3$ $y = -\frac{1}{6}(x + 2)^3$.

---

* For example, if the unit on the y-scale of Fig. 62 be doubled (*i.e.* made equal to the x-scale) while the curve is left unaltered, the graph there given will be the graph of $y = \frac{1}{2}x^3$.

4. Show that the slope of $y = -ax^3$ at the point (x_1, y_1) is $-3ax_1^2$.

5. Discuss the locus of $y = -x^3$.

6. Discuss the locus of $y = -ax^3$ if a is positive and greater than 1; less than 1. Show that the same curve will serve as the graph for all values of $a > 0$ if the units on the axes are properly chosen.

90. The Addition of a Term *mx*. Shearing Motion. If to an expression in x defining a function, a term of the form mx be added, the effect on the graph is readily described in terms of a type of motion that is important in mechanics. For example, let us take the function x^3 and investigate the effect produced upon the graph by adding the term $-3x$. The graphs of $y = x^3$ and $y = -3x$ are drawn in Fig. 65. The graph of $y = x^3 - 3x$ is then obtained by adding the corresponding ordinates of the former graphs. The addition of these two functions is obtained graphically by sliding the ordinate of each point on $y = x^3$ vertically up or down until the base of that ordinate meets the graph of $y = -3x$. If we think of the ordinates of $y = x^3$ as attached to the x-axis and constrained to remain vertical, the graph of $y = x^3$ will become the graph of $y = x^3 - 3x$ if the x-axis is rotated about the origin until it coincides with the line $y = -3x$. The resulting graph of $y = x^3 - 3x$ is, of course, to be interpreted as drawn with reference to the original x-axis. The motion just described, whereby $y = x^3$ is transformed into $y = x^3 - 3x$, is called a ***shearing motion*** or a ***shear*** with respect to $y = -3x$.

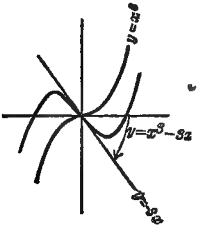


FIG. 65

In general, if the term mx is added to ax^3, the graph of the function $ax^3 + mx$ is obtained by subjecting the graph of ax^3 to a shear with respect to the line $y = mx$. If a and m have the same signs, the effect is in the direction of straightening the graph; if a and m have different signs, the effect is in the direction of emphasizing the curvature.

These effects can be produced by drawing the original figure on the edges of a pack of cards, or on the edges of a book, and then shifting the cards (or sheets of paper) as shown in Fig. 66.

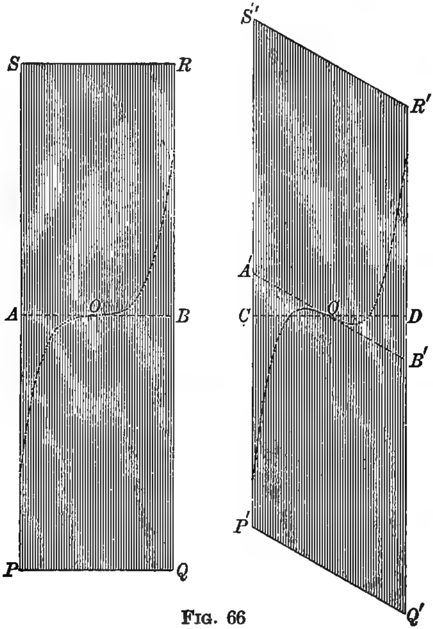


FIG. 66

EXERCISES

Draw the graph of the following functions, making use of the **shear**:

1. $y = 3x^2 + x$.

2. $y = x^2 + x$.

3. $y = -x^3 - x$.

4. $y = -2x^3 + 4x$.

5. $y = x^3 + x - 1$.

6. $y = -x^3 + x + 2$.

7. $y = x^3 - 1$.

8. $y = x^2 - 4x$.

9. Show that $y = mx$ is the equation of the tangent to the curve $y = x^3 + mx$ at the origin.

91. The Functions $a(x-h)^3+m(x-h)+k$ and ax^3+bx^2+cx+d. We have seen that the graph of

$$y = ax^3 + mx \tag{5}$$

has one of the following forms (Fig. 67) according to the signs of a and m.

If such a graph is subjected to a parallel motion which carries the origin to the point (h, k), the equation of the graph in its new position is (§ 78)

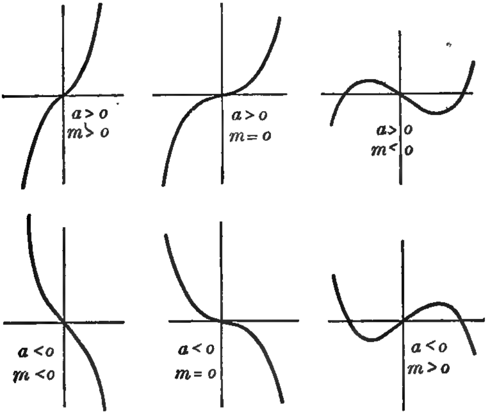


Fig. 67

$$y - k = a(x-h)^3 + m(x-h), \tag{6}$$

which when expanded takes the form

$$y = ax^3 - 3\,ahx^2 + (3\,ah^2 + m)x - ah^3 - mh + k. \tag{7}$$

This is of the general form

$$y = ax^3 + bx^2 + cx + d. \tag{8}$$

Moreover, it includes for all values of h, k, m all the equations of the general form (8). For (7) and (8) will be identical if

$$(9) \qquad -3\,ah = b, \quad 3\,ah^2 + m = c, \quad -ah^3 - mh + k = d.$$

The first of these equations determines $h (a \neq 0)$; h being known, the second equation determines m; m and h being known, the third equation determines k. We may conclude then that the graph of any function of the form (8) has one of the shapes given in Fig. 67, but with the origin moved to a point (h, k) given by the equations (9).

In order to draw the graph of a function of the form (8) we could first transform (8) into the form (6) and then proceed as in § 90. It is more expeditious, however, to proceed more directly by making use of the slope of the function (8) and our knowledge of what shapes may be expected.

92. The Slope of $y = ax^3 + bx^2 + cx + d$. The change Δy in y due to a change Δx in the function

$$y = ax^3 + bx^2 + cx + d,$$

when $x = x_1$, is

$$\Delta y = (3\,ax_1^2 + 2\,bx_1 + c + 3\,ax_1\Delta x + b\Delta x + a\Delta x^2)\,\Delta x.$$

This equation shows that the graph is continuous. Why? When Δx approaches 0, the change ratio $\Delta y/\Delta x$ approaches the slope m, by definition. This gives,

$$m = 3\,ax_1^2 + 2\,bx_1 + c.$$

93. To draw the Graph of $y = ax^3 + bx^2 + cx + d$. We shall illustrate by means of two examples the method of drawing the graph of a cubic function.

Example 1. Draw the graph of $y = x^3 + x^2 - x + 2$.

The slope m at the point (x_1, y_1) is (§ 92)

$$m = 3x_1^2 + 2x_1 - 1.$$

We seek first the points (if such exist) at which the tangent is horizontal, *i.e.* where $m = 0$. The roots of the equation $m = 0$, viz.

$$3x_1^2 + 2x_1 - 1 = 0$$

are $x_1 = -1$ and $x_1 = \frac{1}{3}$. The slope is therefore 0 at the points $(-1, 3)$ and $(\frac{1}{3}, \frac{49}{27})$.

We now compute a table of corresponding values of x, y, m for values of x on both sides of and between $x = 1$ and $x = \frac{1}{3}$. Such a table and the corresponding figure are given below.

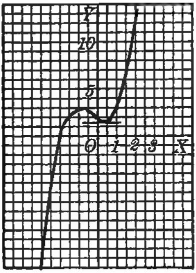


Fig. 68

x	y	m
-3	-13	20
-2	0	7
-1	3	0
0	2	-1
$\frac{1}{3}$	$\frac{49}{27}$	0
1	3	4
2	12	15

Example 2. Draw the graph of $y = -x^3 - 3x^2 - 5x + 1$.

The slope at the point where $x = x_1$ is

$$m = -(3x_1^2 + 6x_1 + 5).$$

Since the roots of the equation $3x_1^2 + 6x_1 + 5 = 0$ are imaginary, the graph has no horizontal tangents and the slope m is negative at every point. We accordingly make a table of values and construct the graph (Fig. 69).

x	y	m
-3	16	-15
-2	7	-5
-1	4	-2
0	1	-5
1	-8	-14

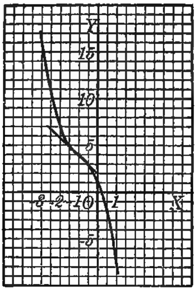


FIG. 69

94. Maxima and Minima. We extend our definition of maximum and minimum given in § 75 as follows:

A value of x for which a function stops increasing and begins to decrease is said to correspond to a *maximum* of the function; a value of x for which the function stops decreasing and begins to increase is said to correspond to a *minimum* of the function. Thus in Ex. 1, § 93 the value $x = -1$ corresponds to the maximum 3 of the function; the value $x = \frac{1}{3}$ corresponds to the minimum $\frac{49}{27}$ of the function.*

EXERCISES

Draw the following curves and locate in each case the maximum and minimum points if there are any:

1. $y = x^3 + x^2.$
2. $y = \frac{x^2}{3} - \frac{5x^2}{2} - 6x + 1.$
3. $y = x^3 - \frac{x^2}{2} - 2x + 1.$
4. $y = x^3 - x^2 - 5x + 2.$
5. $y = 2x^3 + \frac{5x^2}{2} - 4x + 1.$
6. $y = x^3 + x + 1.$
7. $y = x^3.$
8. $y = x^3 - x.$
9. $y = x^3 + 2x^2 + x.$
10. $y = -x^3 - x^2 + x - 1.$

---

* Note that a maximum of a function does not mean the greatest value a function can assume. In Ex. 1, § 93, the value of the function is greater when $x = 2$ than when $x = -1$. It does mean a value of the function which is greater than the values in the immediate neighborhood.

95. Geometric Problems in Maxima and Minima. The theory just explained has an important application in solving problems in maxima and minima, *i.e.* the determination of the largest or the smallest value a magnitude may have which satisfies certain given conditions.

As we saw in § 80, the first step is to express the magnitude in question algebraically. If the resulting expression contains more than one variable, other conditions always will be given which will be sufficient to express all of the variables in terms of one of them. When the magnitude in question is expressed in terms of one variable, we can proceed as in § 92 to find any maximum or minimum values which there may be.

EXAMPLE 1. Find the greatest cylinder that can be cut from a given right circular cone, whose height is equal to the diameter of its base.

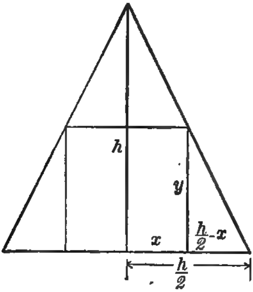


FIG. 70

Let h be the given height of the cone and x and y the unknown dimensions of the cylinder (Fig. 70). Then the volume V of the cylinder is equal to $\pi x^2 y$. But from similar triangles we have

$$\frac{y}{h} = \frac{\frac{h}{2} - x}{\frac{h}{2}}.$$

Therefore,

$$y = h - 2x;$$

whence

$$V = \pi x^2(h - 2x) = \pi h x^2 - 2\pi x^3.$$

Now

$$m = 2\pi h x - 6\pi x^2.$$

The roots of the equation $m = 0$ are $x = 0$ and $x = h/3$.

It is left as an exercise to draw the graph of the function

$$V = \pi h x^2 - 2\pi x^3$$

and show that the value $x = h/3$ corresponds to the *maximum* of the function, *i.e.* to $y = h/3$. Therefore the maximum volume of the cylinder is obtained when the altitude is equal to the radius of the base. The maximum volume is $\pi h^3/27$ or $12/27$ of the volume of the cone.

EXERCISES

1. A square piece of tin, the length of whose side is a, has a small square cut from each corner and the sides are bent up to form a box. Determine the side of the square cut away so that the box shall have the maximum cubical contents. *Ans.* $a/6$.

2. Assuming that the strength of a beam with rectangular cross section varies directly as the breadth and as the square of the depth, what are the dimensions of the strongest beam that can be sawed from a round log whose diameter is d. *Ans.* Depth $= \sqrt{\frac{2}{3}}\, d$.

3. Find the right circular cylinder of greatest volume that can be inscribed in a right circular cone of altitude h and base radius r.

Ans. Radius of the base of the cylinder equals $\frac{2}{3}\, r$.

4. Equal squares are cut from each corner of a rectangular piece of tin 30 inches by 14 inches. Find the side of this square so that the remaining piece of tin will form a box of maximum contents.

5. Show that the maximum and minimum points on the curve $y = x^3 - ax + b$ $(a > 0)$ are at equal distances from the y-axis.

6. Find the maximum volume of a right cone with a given slant height L.

96. The Power Function. The functions x^n and $1/x^n$, where n is any positive integer, are called *power functions* of x. The curves $y = x^n$ (Fig. 71) are known as ***parabolic,*** while the curves $y = 1/x^n$ (Fig. 72) are known as ***hyperbolic.***

The curves of the parabolic type possess the property that they all pass through the point (0, 0) and the point (1, 1).

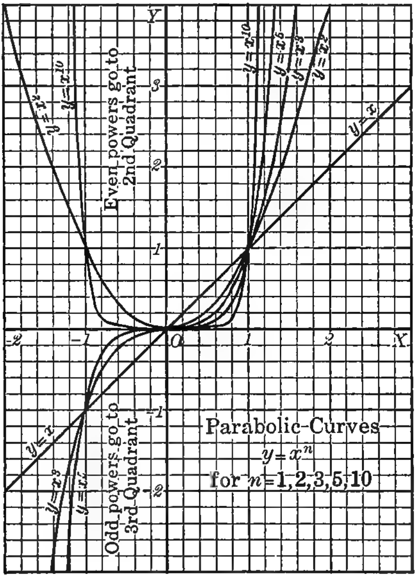


FIG. 71

The larger the value of n, the greater is the slope of the tangent at the point (1, 1).

The curves of the hyperbolic type all pass through the point (1, 1). As x approaches 0, the corresponding value of y becomes infinite. At $x = 0$ the value of y is undefined. As x becomes infinite, the corresponding value of y approaches 0.

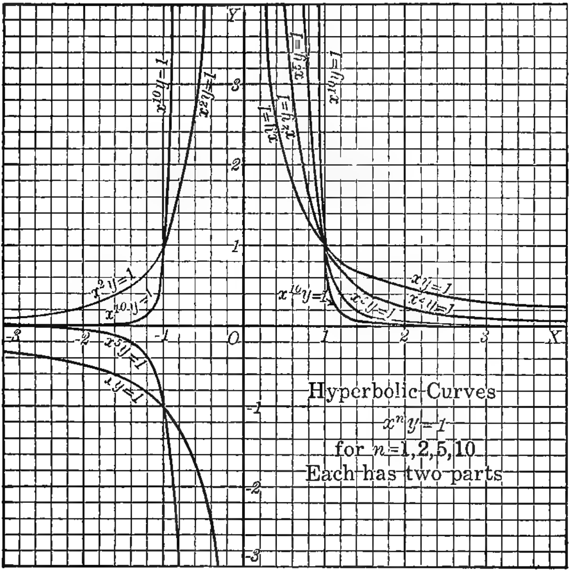


FIG. 72

EXERCISES

1. Draw the curves $y = x^2$; $y = x^3$; $y = x^4$; $y = x^5$.

2. Draw the curves $y = 1/x$; $y = 1/x^2$; $y = 1/x^3$; $y = 1/x^4$.

3. Prove that the slope of the tangent at the point (1, 1) to the curve $y = x^2$, is 2; to the curve $y=x^3$ is 3; to the curve $y=x^4$ is 4; to the curve $y = x^5$ is 5.

4. Prove that for every even value of n, the parabolic curves $y = x^n$ pass through the point $(-1, 1)$; and that for every odd value of n, they pass through the point $(-1, -1)$.

5. Prove that the function x^3 is an increasing function for all values of x.

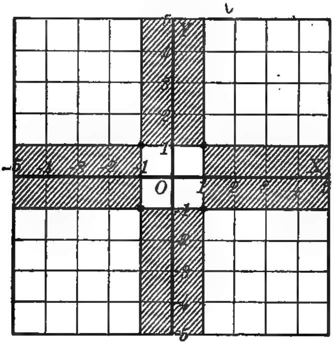
6. Find the equation of the tangent and the normal to $y = x^5$ at the point (2, 32).

7. Prove that the slope of the curve $y = 1/x$ at the point (x_1, y_1) is $-1/x_1^2$. [The curve $y = 1/x$ is called a ***hyperbola.***]

8. How can the graph of the function $y = ax^n$ be obtained from the graph of $y = x^n$ if a is positive? negative?

9. Find the equation of the tangent and the normal to the curve $y = 1/x^2$ at the point $(2, \frac{1}{4})$.

10. Prove that all hyperbolic curves lie within the shaded regions of the adjoining figure, while all parabolic curves lie in the regions left unshaded.



CHAPTER VI

THE TRIGONOMETRIC FUNCTIONS

97. The functions we have discussed hitherto, namely, the functions of the form $mx + b$, $ax^2 + bx + c$, $ax^3 + bx^2 + cx + d$, have all been defined by means of explicit algebraic expressions. They are all examples of a very large class of functions known as ***algebraic functions.*** We now turn our attention to functions defined in an entirely different way. As we shall see, these functions depend on the size of an angle. They enable us to express completely the relations between the sides and the angles of a triangle, and they are of the greatest practical importance in surveying, engineering, and indeed in all branches of pure and applied mathematics.

98. Directed and General Angles. In elementary geometry an angle is usually defined as the figure formed by two half-lines issuing from a point. However, it is often more serviceable to think of an angle as being generated by the rotation in a plane of a half-line OP about the point O as a pivot, starting from the *initial position* OA and ending at the *terminal position* OB (Fig. 73). We then say that the line OP has generated the angle AOB. Similarly, if OP rotates from the initial position OB to the terminal position OA, then the angle BOA is said to be generated. Considerations similar to those regarding directed line segments (§ 6) lead us to regard one of

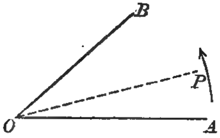


FIG. 73

the above directions of rotation as positive and the other as negative. It is of course quite immaterial which one of the two rotations we regard as positive, but we shall assume from now on, that *counterclockwise rotation* is *positive* and *clockwise rotation* is *negative.*

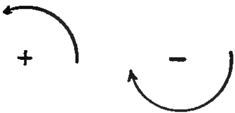


FIG. 74

Still another extension of the notion of angle is desirable. In elementary geometry no angle greater than 360° is considered and seldom one greater than 180°. But from the definition of an angle just given, we see that the revolving line OP may make any number of complete revolutions before coming to rest, and thus the angle generated may be of any magnitude. Angles generated in this way abound in practice and are known as *angles of rotation.*[*]

When the rotation generating an angle is to be indicated, it is customary to mark the angle by means of an arrow starting at the initial line and ending at the terminal line. Unless some such device is used, confusion is liable to result. In Fig. 75 angles of 30°, 390°, 750°, 1110° are drawn. If the angles were not marked one might take them all to be angles of 30°.

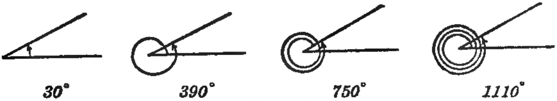


FIG. 75

99. Measurement of Angles. For the present, angles will be measured as in geometry, the degree (°) being the unit of measure. A complete revolution is 360°. The other units in this system are the minute (′), of which 60 make a degree, and the second (″), of which 60 make a minute. This system of units is of great antiquity, having been used by the Babylonians.* The considerations of the previous article then make it clear that any real number, positive or negative, may represent an angle, the absolute value of the number representing the magnitude of the angle, the sign representing the direction of rotation.

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[*] For example, the minute hand of a clock describes an angle of − 180° in 30 minutes, an angle of − 540° in 90 minutes, and an angle of − 720° in 120 minutes.

100. Angles in the Four Quadrants. Consider the angle $XOP = \theta$, whose vertex O coincides with the origin O of a system of rectangular coördinates, and whose initial line OX coincides with the positive half of the x-axis (Fig. 76). The angle θ is then said to be in the first, second, third, or fourth quadrant, according as its terminal line OP is in the first, second, third, or fourth quadrant.

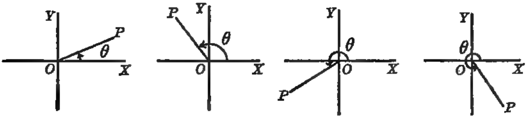

FIG. 76

101. Addition and Subtraction of Directed Angles. The meaning to be attached to the sum of two directed angles is analogous to that for the sum of two directed line segments. Let a and b be two half-lines issuing from the same point O and let (ab) represent an angle obtained by rotating a half-line from the position a to the position b. Then if we have two angles (ab) and (bc) with the same vertex O, the sum $(ab)+(bc)$ of the angles is the angle represented by the rotation of a half-line from the position a to the position b and then rotating from the position b to the position c. But these two rotations are together equivalent to a single rotation from a to c, no matter what the relative positions of a, b, c may have

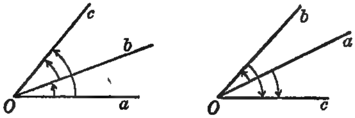

FIG. 77

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* The terms minutes and seconds are derived from their Latin names, which are *partes minutæ primæ* and *partes minutæ secundæ*. At present there is a slight tendency among some authors to divide the degree decimally instead of into minutes and seconds. Still other authors use the degree and minute and divide the minutes decimally. Exercises involving both these systems will be found in the text. When the metric system was introduced at the end of the eighteenth century it was proposed to divide the right angle into 100 parts, called *grades*. The grade was divided into 100 minutes and the minute into 100 seconds. This system is used in some European countries, but not at all in America.

been. Hence, we have for any three half-lines a, b, c issuing from a point O,

(1) $$(ab)+(bc)=(ac), \quad (ab)+(ba)=0, \quad (ab)=(cb)-(ca).$$

The proof of the last relation is left as an exercise.

These relations are analogous to those of § 35; but an essential difference must be noted. Given two points A and B on a line, we may speak of the directed segment AB. The measure of AB is completely determined when A and B and the unit of measure are given. But if the half-lines a and b are given, the angle (ab) may be any angle generated by a rotation from a to b. Such angles may be positive or negative and may involve, in addition to the minimum rotation from a to b, any number of complete revolutions. It is to be noted, however, that *all possible determinations of the angle* (ab) *differ among themselves only by integral multiples of* 360°. In other words, if θ represents the smallest positive measure (in degrees) of an angle from a to b, then any determination of (ab) is given by the relation $(ab) = \theta \pm n \cdot 360°$ (n an integer). The equality signs in relations (1) are then to be interpreted as meaning *equal, except for multiples of* 360°.

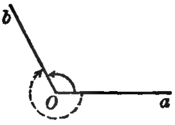

FIG. 78

If the position of the half-line l_1 is determined by the angle θ_1 which it makes with a given horizontal line OX, and the position of another half-line l_2 is determined by the angle θ_2 which it makes with OX we have

$$\text{angle from } l_1 \text{ to } l_2 = \theta_2 - \theta_1,$$

except for multiples of 360°. Why?

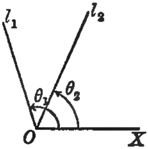

FIG. 79

EXERCISES

1. What angle does the minute hand of a clock describe in 2 hours and 30 minutes? in 4 hours and 20 minutes?

2. Suppose that the dial of a clock is transparent so that it may be read from both sides. Two persons stationed on opposite sides of the dial observe the motion of the minute hand. In what respect will the angles described by the minute hand as seen by the two persons differ?

3. In what quadrants are the following angles: 87°? 135°? – 325°? 540°? 1500°? – 270°?

4. In what quadrant is $\theta/2$ if θ is a positive angle less than 360° and in the second quadrant? third quadrant? fourth quadrant?

5. By means of a protractor construct $27° + 85° + (-30°) + 20° + (-45°)$.

6. By means of a protractor construct $-130° + 56° - 24°$.

102. The Sine, Cosine, and Tangent of an Angle. We may now define three of the functions referred to in § 97. To this end let $\theta = XOP$ (Fig. 80) be any directed angle, and let us establish a system of rectangular coördinates in the plane of the angle such that the initial side OX of the angle is the positive half of the x-axis, the vertex O being at the origin and the y-axis being in the usual position with respect to the x-axis. Let the units on the two axes be equal. Finally, let P be any point other than O on the terminal side of the angle θ, and let its coördinates be (x, y). The directed segment $OP = r$ is called the *distance of* P and is always chosen positive. The coördinates x and y are positive or negative according to the conventions previously adopted. We then define

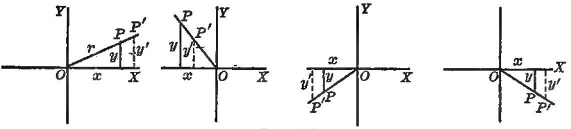

FIG. 80

$$\textit{The sine of } \theta = \frac{\text{ordinate of } P}{\text{distance of } P} = \frac{y}{r},$$

$$\textit{The cosine of } \theta = \frac{\text{abscissa of } P}{\text{distance of } P} = \frac{x}{r},$$

$$\textit{The tangent of } \theta = \frac{\text{ordinate of } P}{\text{abscissa of } P} = \frac{y}{x}, \text{ provided } x \neq 0,^*$$

These functions are usually written in the abbreviated forms $\sin\theta$, $\cos\theta$, $\tan\theta$, respectively; but they are read as "sine θ," "cosine θ," "tangent θ." It is very important to notice that *the values of these functions are independent of the position of the point P on the terminal line.* For let $P'(x', y')$ be any other point on this line. Then from the similar right triangles

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* Prove that x and y cannot be zero simultaneously.

xyr[*] and $x'y'r'$ it follows that the ratio of any two sides of the triangle xyr is equal in magnitude and sign to the ratio of the corresponding sides of the triangle $x'y'r'$. Therefore the values of the functions just defined depend merely on the angle θ. They are one-valued functions of θ and are called ***trigonometric functions.***[†]

Since the values of these functions are defined as the ratio of two directed segments, they are abstract numbers. They may be either positive, negative, or zero. Remembering that r is always positive, we may readily verify that the signs of the three functions are given by the following table.

Quadrant	1	2	3	4
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent	+	−	+	−

103. Values of the Functions for 45°, 135°, 225°, 315°. In each of these cases the triangle xyr is isosceles. Why? Since the trigonometric functions are independent of the position of the point P on the terminal line, we may choose the legs of the right triangle xyr to be of length unity, which

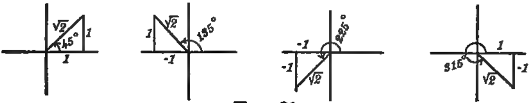

FIG. 81

gives the distance OP as $\sqrt{2}$. Figure 81 shows the four angles with all lengths and directions marked. Therefore,

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[*] Triangle xyz means the triangle whose sides are x, y, z.

[†] *Trigonometric* etymologically means *relating to the measurement of triangles.* The connection of these functions with triangles will appear presently.

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}, \qquad \tan 45^\circ = 1,$$

$$\sin 135^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 135^\circ = -\frac{1}{\sqrt{2}}, \qquad \tan 135^\circ = -1,$$

$$\sin 225^\circ = -\frac{1}{\sqrt{2}}, \qquad \cos 225^\circ = -\frac{1}{\sqrt{2}}, \qquad \tan 225^\circ = 1,$$

$$\sin 315^\circ = -\frac{1}{\sqrt{2}}, \qquad \cos 315^\circ = \frac{1}{\sqrt{2}}, \qquad \tan 315^\circ = -1.$$

104. Values of the Functions for 30°, 150°, 210°, 330°. From geometry we know that if one angle of a right triangle contains 30°, then the hypotenuse is double the shorter leg, which is opposite the 30° angle. Hence if we choose the shorter leg (ordinate) as 1, the hypotenuse (distance) is 2, and the other leg (abscissa) is $\sqrt{3}$. Figure 82 shows angles of 30°, 150°, 210°, 330° with all lengths and directions marked. Hence we have

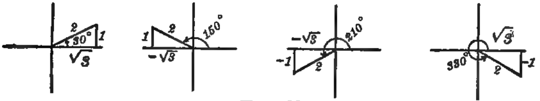

FIG. 82

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}},$$

$$\sin 150^\circ = \frac{1}{2}, \qquad \cos 150^\circ = -\frac{\sqrt{3}}{2}, \qquad \tan 150^\circ = -\frac{1}{\sqrt{3}},$$

$$\sin 210^\circ = -\frac{1}{2}, \qquad \cos 210^\circ = -\frac{\sqrt{3}}{2}, \qquad \tan 210^\circ = \frac{1}{\sqrt{3}},$$

$$\sin 330^\circ = -\frac{1}{2}, \qquad \cos 330^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 330^\circ = -\frac{1}{\sqrt{3}}.$$

105. Values of the Functions for 60°, 120°, 240°, 300°. It is left as an exercise to construct these angles and to prove that

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \tan 60^\circ = \sqrt{3},$$

$$\sin 120^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 120^\circ = -\frac{1}{2}, \qquad \tan 120^\circ = -\sqrt{3},$$

$$\sin 240^\circ = -\frac{\sqrt{3}}{2}, \qquad \cos 240^\circ = -\frac{1}{2}, \qquad \tan 240^\circ = \sqrt{3},$$

$$\sin 300^\circ = -\frac{\sqrt{3}}{2}, \qquad \cos 300^\circ = \frac{1}{2}, \qquad \tan 300^\circ = -\sqrt{3}.$$

106. Applications. The angle which a line from the eye to an object makes with a horizontal line in the same vertical plane is called an *angle of elevation* or an *angle of depression*

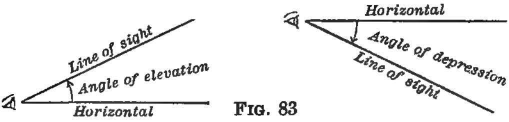

FIG. 83

according as the object is above or below the eye of the observer (Fig. 83). Such angles occur in many examples.

EXAMPLE 1. A man wishing to know the distance between two points A and B on opposite sides of a pond, locates a point C on the land (Fig. 84) such that $AC = 200$ rd., angle $C = 30°$, and angle $B = 90°$. Find the distance AB.

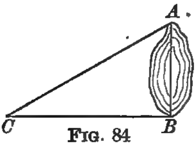

FIG. 84

SOLUTION :

$$\frac{AB}{AC} = \sin C. \qquad \text{(Why ?)}$$

$$\begin{aligned} AB &= AC \cdot \sin C \\ &= 200 \cdot \sin 30^\circ \\ &= 200 \cdot \tfrac{1}{2} = 100 \text{ rd.} \end{aligned}$$

EXAMPLE 2. Two men stationed at points A and C 800 yd. apart and in the same vertical plane with a balloon B, observe simultaneously the angle of elevation of the balloon to be 30° and 45° respectively. Find the height of the balloon.

Solution: Denote the height of the balloon DB by y, and let $DC = x$; then $AD = 800 - x$.

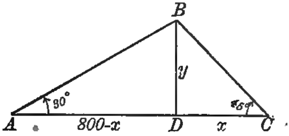

FIG. 85

Since $\tan 45^\circ = 1$, we have $1 = \dfrac{y}{x}$

and since $\tan 30^\circ = 1/\sqrt{3}$, we have $\dfrac{1}{\sqrt{3}} = \dfrac{y}{800 - x}$.

Therefore $x = y$ and $800 - x = y\sqrt{3}$.

Solving these equations for y, we have $y = \dfrac{800}{\sqrt{3} + 1} = 292.8$ yd.

EXERCISES

1. In what quadrants is the sine positive? cosine negative? tangent positive? cosine positive? tangent negative? sine negative?

2. In what quadrant does an angle lie if

(*a*) its sine is positive and its cosine is negative?
(*b*) its tangent is negative and its cosine is positive?
(*c*) its sine is negative and its cosine is positive?
(*d*) its cosine is positive and its tangent is positive?

3. Which of the following is the greater and why: $\sin 49^\circ$ or $\cos 49^\circ$? $\sin 35^\circ$ or $\cos 35^\circ$?

4. If θ is situated between 0° and 360°, how many degrees are there in θ if $\tan\theta = 1$? Answer the similar question for $\sin\theta = \frac{1}{2}$; $\tan\theta = -1$.

5. Does $\sin 60^\circ = 2 \cdot \sin 30^\circ$? Does $\tan 60^\circ = 2 \cdot \tan 30^\circ$? What can you say about the truth of the equality $\sin 2\theta = 2 \sin\theta$?

6. The Washington Monument is 555 ft. high. At a certain place in the plane of its base, the angle of elevation of the top is 60°. How far is that place from the foot and from the top of the tower?

7. A boy whose eyes are 5 ft. from the ground stands 200 ft. from a flagstaff. From his eyes, the angle of elevation of the top is 30°. How high is the flagstaff?

8. A tree 38 ft. high casts a shadow 38 ft. long. What is the angle of elevation of the top of the tree as seen from the end of the shadow? How far is it from the end of the shadow to the top of the tree?

9. From the top of a tower 100 ft. high, the angle of depression of two stones, which are in a direction due east and in the plane of the base, are 45° and 30° respectively. How far apart are the stones?

Ans. $100(\sqrt{3}-1)=73.2$ ft.

10. Find the area of the isosceles triangle in which the equal sides 10 inches in length include an angle of 120°. *Ans.* $25\sqrt{3}=43.3$ sq. in.

11. Is the formula $\sin 2\theta = 2\sin\theta\cos\theta$ true when $\theta = 30°$? $60°$? $120°$?

12. From a figure prove that $\sin 117° = \cos 27°$.

13. Find the tangent of the angle which the line joining the points (x_1, y_1), and (x_2, y_2) makes with the x-axis, assuming the units on the two axes to be equal. Compare your answer with the definition of slope in §§ 50 and 53.

14. Determine whether each of the following formulas is true when $\theta = 30°, 60°, 150°, 210°$:

$$1+\tan^2\theta = \frac{1}{\cos^2\theta},$$

$$1+\frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta},$$

$$\sin^2\theta + \cos^2\theta = 1.$$

15. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points the distance between which is r (the units on the axes being equal). If θ is the angle that the line P_1P_2 makes with the x-axis, prove that

$$\frac{x_2-x_1}{\cos\theta}+\frac{y_2-y_1}{\sin\theta}=2r.$$

107. Computation of the Value of One Trigonometric Function from that of Another.

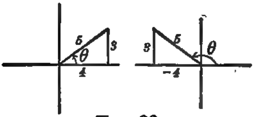

FIG. 86

Example 1. Given that $\sin\theta = \frac{3}{5}$, find the values of the other functions.

Since $\sin\theta$ is positive, it follows that θ is an angle in the first or in the second quadrant. Moreover, since the value of the sine is $\frac{3}{5}$, then $y = 3\cdot k$ and $r = 5\cdot k$, where k is any positive constant different from zero. (Why?) It is, of course, immaterial what positive value we assign to k, so we shall assign the value 1. We know, however, that the abscissa, ordinate, and distance are connected by the relation $x^2 + y^2 = r^2$, and hence it follows that $x = \pm 4$. Fig. 86 is then self-explanatory. Hence we have, for the first quadrant, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, and $\tan\theta = \frac{3}{4}$; for the second quadrant, $\sin\theta = \frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$.

EXAMPLE 2. Given that $\sin\theta = \frac{5}{13}$ and that $\tan\theta$ is negative, find the other trigonometric functions of the angle θ.

Since $\sin\theta$ is positive and $\tan\theta$ is negative, θ must be in the second quadrant. We can, therefore, construct the angle (Fig. 87), and we obtain $\sin\theta = \frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\tan\theta = -\frac{5}{12}$.

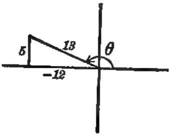

FIG. 87

108. Computation for Any Angle. Tables. The values of the trigonometric functions of any angle may be computed by the graphic method. For example, let us find the trigonometric functions of 35°. We first construct on square ruled paper, by means of a protractor, an angle of 35° and choose a point P on the terminal line so that OP shall equal 100 units. Then from the figure we find that $OM = 82$ units and $MP = 57$ units. Therefore

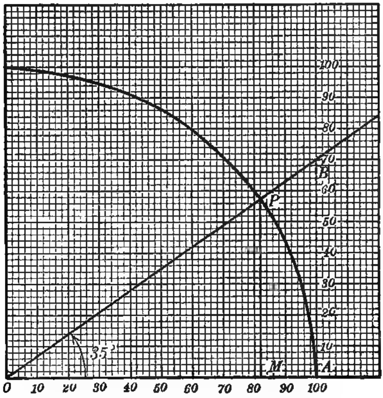

FIG. 88

$$\sin 35° = \tfrac{57}{100} = 0.57,\ \cos 35° = \tfrac{82}{100} = 0.82,\ \tan 35° = \tfrac{57}{82} = 0.70.$$

The tangent may be found more readily if we start by taking $OA = 100$ units and then measure AB. In this case, $AB = 70$ units and hence $\tan 35° = \frac{70}{100} = 0.70$.

It is at once evident that the graphic method, although

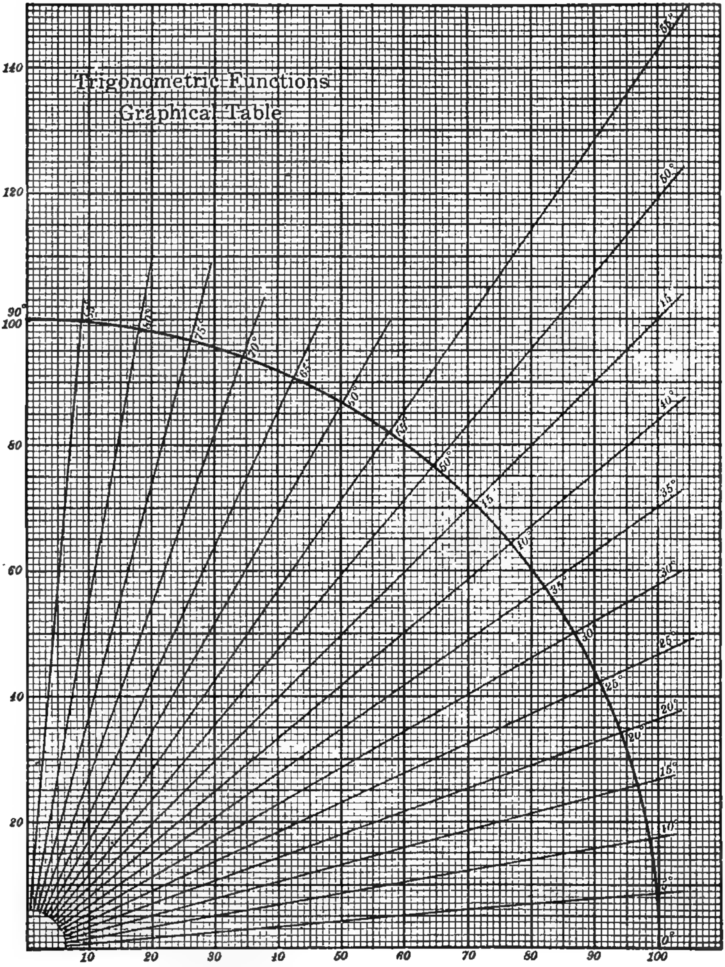

FIG. 89. — GRAPHICAL TABLE OF TRIGONOMETRIC FUNCTIONS

simple, gives only an approximate result. However, the values of these functions have been computed accurately by methods beyond the scope of this book. The results have been put in tabular form and are known as tables of natural trigonometric functions. These tables with an explanation of their use will be found in any good set of mathematical tables.* In order to solve several of the following exercises it is necessary to make use of such tables.

Figure 89 makes it possible to read off the sine, cosine, or tangent of any angle between 0° and 90° with a fair degree of accuracy. The figure is self-explanatory. Its use is illustrated in some of the following exercises.

EXERCISES

Find the other trigonometric functions of the angle θ when

1. $\tan\theta = -3$.

2. $\sin\theta = -\frac{3}{5}$.

3. $\cos\theta = \frac{12}{13}$.

4. $\tan\theta = \frac{4}{7}$.

5. $\sin\theta = \frac{3}{4}$.

6. $\cos\theta = -\frac{1}{5}$

7. $\sin\theta = \frac{3}{5}$ and $\cos\theta$ is negative.

8. $\tan\theta = 2$ and $\sin\theta$ is negative.

9. $\sin\theta = -\frac{1}{4}$ and $\tan\theta$ is positive.

10. $\cos\theta = \frac{3}{5}$ and $\tan\theta$ is negative.

11. Can 0.6 and 0.8 be the sine and cosine, respectively, of one and the same angle? Can 0.5 and 0.9? *Ans.* Yes; no.

12. Is there an angle whose sine is 2? Explain.

13. Determine graphically the functions of 20°, 38°, 70°, 110°. Check your results by the tables of natural functions.

14. From Fig. 89, find values of the following:

sin 10°, cos 50°, tan 40°, sin 80°, tan 70°, cos 32°, tan 14°, sin 14°.

15. A tower stands on the shore of a river 200 ft. wide. The angle of elevation of the top of the tower from the point on the other shore exactly opposite to the tower is such that its sine is $\frac{3}{5}$. Find the height of the tower.

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* See, for example, THE MACMILLAN TABLES, which will be referred to in connection with this book.

16. From a ship's masthead 160 feet above the water the angle of depression of a boat is such that the tangent of this angle is $\frac{1}{12}$. Find the distance from the boat to the ship. *Ans.* 640 yards.

17. A certain railroad rises 6 inches for every 10 feet of track. What angle does the track make with the horizontal?

18. On opposite shores of a lake are two flagstaffs A and B. Perpendicular to the line AB and along one shore, a line $BC = 1200$ ft. is measured. The angle ACB is observed to be $40° 20'$. Find the distance between the two flagstaffs.

19. The angle of ascent of a road is 8°. If a man walks a mile up the road, how many feet has he risen?

20. How far from the foot of a tower 150 feet high must an observer, 6 ft. high, stand so that the angle of elevation of its top may be $23°.5$?

21. From the top of a tower the angle of depression of a stone in the plane of the base is $40° 20'$. What is the angle of depression of the stone from a point halfway down the tower?

22. The altitude of an isosceles triangle is 24 feet and each of the equal angles contain $40° 20'$. Find the lengths of the sides and area of the triangle.

23. A flagstaff 21 feet high stands on the top of a cliff. From a point on the level with the base of the cliff, the angles of elevation of the top and bottom of the flagstaff are observed. Denoting these angles by α and β respectively, find the height of the cliff in case $\sin\alpha = \frac{8}{17}$ and $\cos\beta = \frac{12}{13}$. *Ans.* 75 feet.

24. A man wishes to find the height of a tower CB which stands on a horizontal plane. From a point A on this plane he finds the angle of elevation of the top to be such that $\sin CAB = \frac{3}{5}$. From a point A' which is on the line AC and 100 feet nearer the tower, he finds the angle of elevation of the top to be such that $\tan CA'B = \frac{3}{2}$. Find the height of the tower.

25. Find the radius of the inscribed and circumscribed circle of a regular pentagon whose side is 14 feet.

26. If a chord of a circle is two thirds of the radius, how large an angle at the center does the chord subtend?

27. A boy standing a feet behind and opposite the middle of a football goal observes the angle of elevation of the nearer crossbar to be α, and the angle of elevation of the farther crossbar to be β. Prove that the length of the field is $a[\tan\alpha - \tan\beta]/\tan\beta$.

109. The Sine Function. Let us trace in a general way the variation of the function $\sin\theta$ as θ increases from 0° to 360°. For this purpose it will be convenient to think of the distance r as constant, from which it follows that the locus of P is a circle. When $\theta = 0°$, the point P lies on the x-axis and hence the ordinate is 0, *i.e.* $\sin 0° = 0/r = 0$. As θ increases to 90°, the ordinate increases until 90° is reached, when it becomes equal to r. Therefore, $\sin 90° = r/r = 1$. As θ increases from 90° to 180°, the ordinate decreases until 180° is reached, when it becomes 0. Therefore $\sin 180° = 0/r = 0$. As θ increases from 180° to 270°, the ordinate of P continually decreases algebraically and reaches its smallest algebraic value when $\theta = 270°$. In this position the ordinate is $-r$ and $\sin 270° = -r/r = -1$. When θ enters the fourth quadrant, the ordinate of P increases (algebraically) until the angle reaches 360°, when the ordinate becomes 0. Hence, $\sin 360° = 0$. It then appears that:

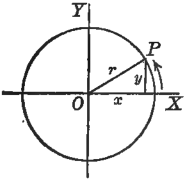

FIG. 90

as θ increases from 0° to 90°, $\sin\theta$ increases from 0 to 1;
as θ increases from 90° to 180°, $\sin\theta$ decreases from 1 to 0;
as θ increases from 180° to 270°, $\sin\theta$ decreases from 0 to -1;
as θ increases from 270° to 360°, $\sin\theta$ increases from -1 to 0.

It is evident that the function $\sin\theta$ repeats its values in the same order no matter how many times the point P moves around the circle. We express this fact by saying that the function $\sin\theta$ is **periodic** and has a *period* of 360°. In symbols this is expressed by the equation

$$\sin[\theta + n \cdot 360°] = \sin\theta,$$

where n is any positive or negative integer.

The variation of the function $\sin\theta$ is well shown by its graph. To construct this graph proceed as follows: Take a system of rectangular axes and construct a circle of unit radius with its center on the x-axis (Fig. 91). Let angle $XOP = \theta$. Then the values of $\sin\theta$ for certain values of θ are shown in the unit circle as the ordinates of the end of the radius drawn at an angle θ.

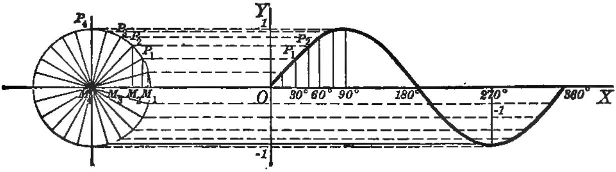

FIG. 91

θ	0	30°	45°	60°	90°	⋯
$\sin\theta$	0	M_1P_1	M_2P_2	M_3P_3	M_4P_4	⋯

Now let the number of degrees in θ be represented by distances measured along OX. At a distance that represents 30° erect a perpendicular equal in length to $\sin 30°$; at a distance that represents 60° erect one equal in length to $\sin 60°$, etc. Through the points O, P_1, P_2, $\cdots$ draw a smooth curve; this curve is the graph of the function $\sin\theta$.

If from any point P on this graph a perpendicular PQ is drawn to the x-axis, then QP represents the sine of the angle represented by the segment OQ.

Since the function is periodic, the complete graph extends indefinitely in both directions from the origin (Fig. 92).

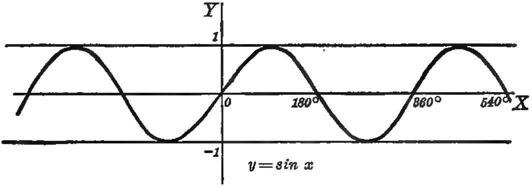

FIG. 92

110. The Cosine Function. By arguments similar to those used in the case of the sine function we may show that:
as θ increases from 0° to 90°, the $\cos\theta$ decreases from 1 to 0;
as θ increases from 90° to 180°, the $\cos\theta$ decreases from 0 to -1;
as θ increases from 180° to 270°, the $\cos\theta$ increases from -1 to 0;
as θ increases from 270° to 360°, the $\cos\theta$ increases from 0 to 1.

The graph of the function is readily constructed by a method similar to that used in case of the sine function. This is illustrated in Fig. 93.

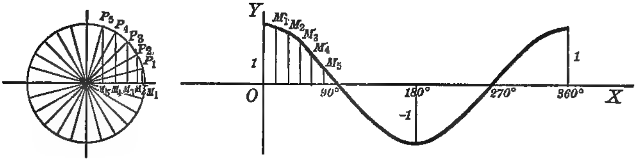

FIG. 93

The complete graph of the cosine function, like that of the sine function, will extend indefinitely from the origin in both directions (Fig. 94). Moreover $\cos\theta$, like $\sin\theta$, is *periodic* and has a *period* of 360°, *i.e.*

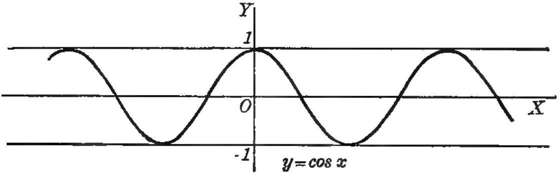

FIG. 94

$$\cos[\theta + n \cdot 360°] = \cos\theta,$$

where n is any positive or negative integer.

111. The Tangent Function. In order to trace the variation of the tangent function, consider a circle of unit radius with its center at the origin of a system of rectangular axes (Fig. 95). Then construct the tangent to this circle at the point $M(1, 0)$ and let P denote any point on this tangent line. If angle $MOP = \theta$, we have $\tan\theta = MP/OM = MP/1 = MP$, *i.e.* the line MP represents $\tan\theta$.

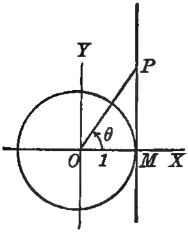

FIG. 95

Now when $\theta = 0°$, MP is 0, *i.e.* tan 0° is 0. As the angle θ increases, $\tan\theta$ increases. As θ approaches 90° as a limit, MP becomes infinite, *i.e.* $\tan\theta$ becomes larger than any number whatever. *At* 90° *the tangent is undefined.* It is sometimes convenient to express this fact by writing

$$\tan 90° = \infty.$$

However we must remember that this is *not a definition* for tan 90°, for ∞ is not a number. This is merely a short way of saying that *as θ approaches* 90°, *tan θ becomes infinite* and that at 90° $\tan\theta$ is undefined. See § 36.

Thus far we have assumed θ to be an acute angle approaching 90° as a limit. Now let us start with θ as an obtuse angle and let it decrease towards 90° as a limit. In Fig. 96 the line MP' (which is here negative in direction) represents $\tan\theta$. Arguing precisely as we did before, it is seen that as the angle θ approaches 90° as a limit, $\tan\theta$ again increases in magnitude beyond all bounds, *i.e.* becomes infinite, remaining, however, always negative.

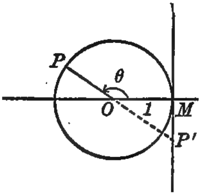

FIG. 96

We then have the following results.

(1) When θ is acute and increases toward 90° as a limit, $\tan\theta$ always remains positive but becomes infinite. At 90° $\tan\theta$ is undefined.

(2) When θ is obtuse and decreases towards 90° as a limit, $\tan\theta$ always remains negative but becomes infinite. At 90° $\tan\theta$ is undefined.

It is left as an exercise to finish tracing the variation of the tangent function as θ varies from 90° to 360°. Note that $\tan 270°$, like $\tan 90°$, is undefined. In fact $\tan n \cdot 90°$ is undefined, if n is any odd integer.

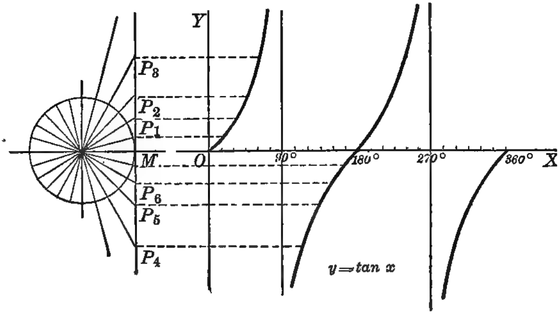

FIG. 97

To construct the graph of the function $\tan\theta$ we proceed along lines similar to those used in constructing the graph of $\sin\theta$ and $\cos\theta$. The following table together with Fig. 97 illustrates the method.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°	210°
$\tan\theta$	0	MP_1	MP_2	MP_3	undefined	MP_4	MP_5	MP_6	$MP_7 = 0$	MP_1

It is important to notice that $\tan\theta$, like $\sin\theta$ and $\cos\theta$, is *periodic*, but its *period* is 180°. That is

$$\tan(\theta + n\cdot 180°) = \tan\theta,$$

where n is any positive or negative integer.

EXERCISES

1. What is meant by the period of a trigonometric function?

2. What is the period of $\sin\theta$? $\cos\theta$? $\tan\theta$?

3. Is $\sin\theta$ defined for all angles? $\cos\theta$?

4. Explain why $\tan\theta$ is undefined for certain angles. Name four angles for which it is undefined. Are there any others?

5. Is $\sin(\theta + 360°) = \sin\theta$?

6. Is $\sin(\theta + 180°) = \sin\theta$?

7. Is $\tan(\theta + 180°) = \tan\theta$?

8. Is $\tan(\theta + 360°) = \tan\theta$?

Draw the graphs of the following functions and explain how from the graph you can tell the period of the function:

9. $\sin\theta$.

10. $\cos\theta$.

11. $\tan\theta$.

12. $\dfrac{1}{\sin\theta}$.

13. $\dfrac{1}{\cos\theta}$.

14. $\dfrac{1}{\tan\theta}$.

Verify the following statements:

15. $\sin 90° + \sin 270° = 0$.

16. $\cos 90° + \sin 0° = 0$.

17. $\tan 180° + \cos 180° = -1$.

18. $\cos 180° + \sin 180° = -1$.

19. $\tan 360° + \cos 360° = 1$.

20. $\cos 90° + \tan 180° - \sin 270° = 1$.

21. Draw the graphs of the functions $\sin\theta$, $\cos\theta$, $\tan\theta$, making use of a table of natural functions. See p. 538.

22. Draw the curves $y = 2\sin\theta$; $y = 2\cos\theta$; $y = 2\tan\theta$.

23. Draw the curve $y = \sin\theta + \cos\theta$.

24. From the graphs determine values of θ for which $\sin\theta = \frac{1}{2}$; $\sin\theta = 1$; $\tan\theta = 1$; $\cos\theta = \frac{1}{2}$; $\cos\theta = 1$.

112. Polar Coördinates. It is convenient at this point to introduce a new way of locating the position of a point in a plane, and of representing the graph of a function. To this end (Fig. 98) let OA be a directed line in the plane which we shall call the ***initial line*** or the ***polar axis***. This line is usually drawn horizontally and directed to the right. The point O is called the ***pole*** or the ***origin***. Let P be any point in the plane and draw the line OP. The position of P is then located completely if we know the angle $AOP = \theta$ and the distance $OP = \rho$. The two numbers (ρ, θ), called respectively the ***radius vector*** and the ***vectorial angle***, are known as the ***polar coördinates*** of the point P.

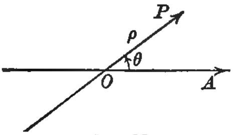

FIG. 98

In Fig. 98 we have represented a case in which θ and ρ are both positive. Either θ or ρ or both may be negative under the following conventions. The angle θ is positive or negative according to the direction of its rotation, as in § 98. The positive direction on OP is the direction from O along the terminal side of the angle θ, *i.e.*, it is the direction into which OA is rotated by a rotation through the angle θ.

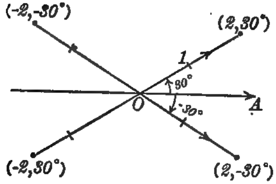

FIG. 99

With these conventions a point P whose polar coördinates (ρ, θ) are given is completely determined. Figure 99 shows points whose polar coördinates are $(2, 30°)$, $(-2, 30°)$, $(2, -30°)$, and $(-2, -30°)$. It will be noted that, if ρ is positive, P is on the terminal side of θ, while if ρ is negative, P is on the terminal side produced through O. On the other hand, a given point P has an unlimited number of polar coördinates (ρ, θ). Even if we confine ourselves to angles in absolute value less than 360°, a point has in general *four* different sets of polar coördinates. Fig. 100 shows that the same

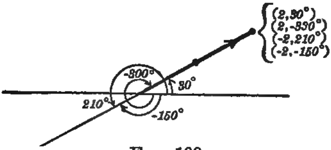

FIG. 100

point P may be designated by any one of the pairs of values (2, 30°), (2, – 330°), (– 2, 210°), and (– 2, – 150°).

EXERCISES

1. Locate the points whose polar coördinates have the following values: (4, 30°), (– 2, 45°), (– 3, – 60°), (2, – 150°), (3, – 90°), (2, 180°), (– 2, 0°), (0, 90°), (– 2, 180°), (– 3, 270°).

2. For each of the points in Ex. 1, give all other sets of polar coördinates for which θ is in absolute value less than 360°.

3. What exceptions are there to the statement "θ being confined to angles in absolute value less than 360°, every point has four and only four distinct sets of polar coördinates"?

4. Where are all the points for which θ is a given constant?

5. Where are all the points for which ρ is a given constant?

113. Graphs in Polar Coördinates. Polar coördinates may be used to represent the graph of a given function, in a way quite similar to that in the case of rectangular coördinates. Fig. 101 gives an example in which the idea of polar coördinates is used in practice. In this example the θ-scale represents *time*, the ρ-scale represents *temperature.*[*] Some forms of self-recording hygrometers employ the same idea.

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[*] It will be noted that in this example the radius vector is measured along a circular arc instead of along a straight line. This is due to the mechanical construction of the instrument. Cf. footnote, p. 9. The fundamental idea is, nevertheless, that of polar coördinates.

In *plotting the graph of a function* in polar coördinates we proceed as in the case of rectangular coördinates. A table of

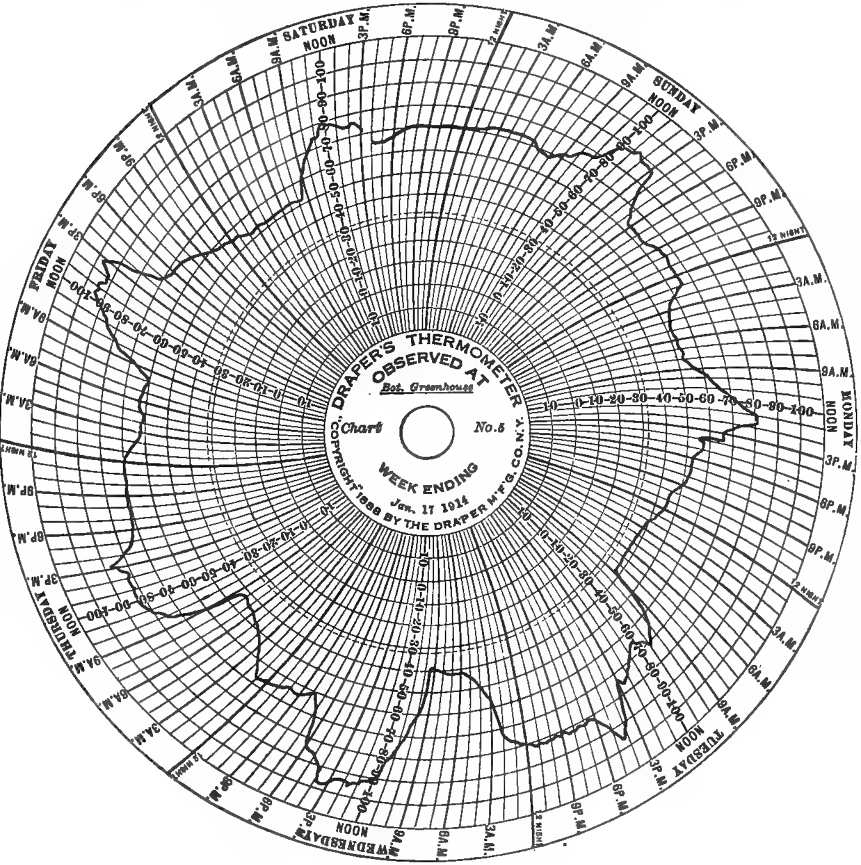


FIG. 101

corresponding values of the variable θ and the function ρ is constructed. Each such pair of values is then plotted as a point, and a curve drawn through these points.

Example. Plot in polar coördinates the graph of $\rho = \sin\theta$. We obtain the table below. Figure 102 exhibits the corresponding points, with a curve drawn through them. Observe that each point serves to represent two pairs of corresponding values. Thus the pairs $(\frac{1}{2}, 30°)$ and $(-\frac{1}{2}, 210°)$ are represented by the same point. This curve suggests a circle, of diameter unity, tangent to the polar axis at the origin.

θ	$\rho = \sin\theta$
0°	.00
30°	.50
45°	.71
60°	.87
90°	1.00
120°	.87
135°	.71
150°	.50
180°	.00
210°	− .50
225°	− .71
240°	− .87
270°	− 1.00
300°	− .87
315°	− .71
330°	− .50
360°	.00

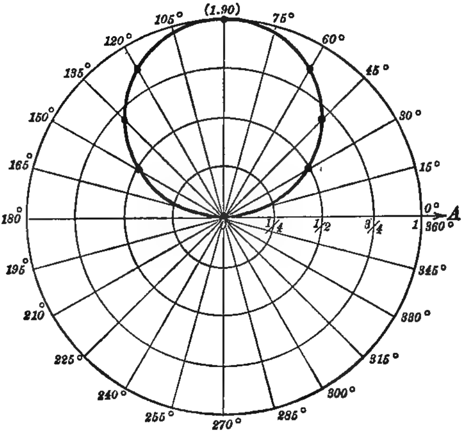


Fig. 102

114. The Graph of sin θ and cos θ in Polar Coördinates.

We may now prove:

The graph, in polar coördinates, of the function $\rho = \sin\theta$ *is a circle of diameter unity, tangent to the polar axis at the origin.*

Let $P(\rho, \theta)$ be any point on such a circle (Fig. 103). Then, for any value θ in the first quadrant

$$\frac{OP}{OA} = \frac{\rho}{1} = \sin\theta \quad \text{or} \quad \rho = \sin\theta.$$

Conversely, if $\rho = \sin\theta$, the point P is on the circle. Why? A similar proof, which is left as an exercise, may be given when θ is in the second, third, or fourth quadrants (Fig. 104). Similarly, we may prove:

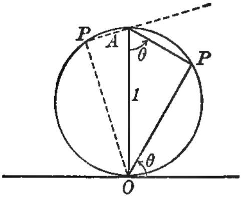


FIG. 103

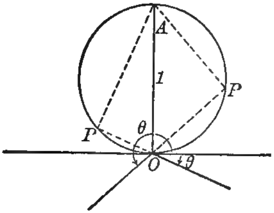


FIG. 104

The graph of

$$\rho = \cos\theta$$

in polar coördinates is a circle of diameter unity, passing through the pole and having its center on the polar axis.

The proof of this statement is left as an exercise. See Figs. 105, 106.

On account of their simplicity, the polar graphs of $\sin\theta$ and $\cos\theta$ are very serviceable. It is for this reason that we have introduced them at this point. Polar coördinates will be discussed again, particularly in Chapter XIV, and incidentally in other chapters.

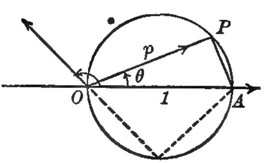


FIG. 105

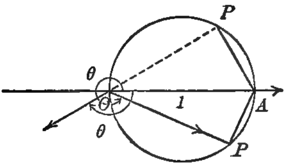


FIG. 106

EXERCISES

1. From Fig. 101, find the temperature at 9 P.M. on Tuesday; at 3 P.M. on Monday. When was the temperature a maximum? a minimum?

2. Plot in polar coördinates the graph representing the variation in temperature given in Ex. 1, p. 16.

3. Plot the graph in polar coördinates of the function $\rho = \tan \theta$. Why is this graph not convenient to represent the function $\tan \theta$?

4. Prove that the graph, in polar coördinates, of $\rho = a \cos \theta$ is a circle of diameter a, passing through the origin and with its center on the polar axis.

5. Prove a theorem regarding the graph of $\rho = a \sin \theta$.

115. Other Trigonometric Functions. The reciprocals of the sine, the cosine, and the tangent of any angle are called, respectively, the cosecant, the secant, and the cotangent of that angle. Thus,

$$\text{cosecant } \theta = \frac{\text{distance of } P}{\text{ordinate of } P} = \frac{r}{y} \quad (\text{provided } y \neq 0).$$

$$\text{secant } \theta = \frac{\text{distance of } P}{\text{abscissa of } P} = \frac{r}{x} \quad (\text{provided } x \neq 0).$$

$$\text{cotangent } \theta = \frac{\text{abscissa of } P}{\text{ordinate of } P} = \frac{x}{y} \quad (\text{provided } y \neq 0).$$

These functions are written csc θ, sec θ, ctn θ. From the definitions follow directly the relations

$$\csc \theta = \frac{1}{\sin \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \text{ctn } \theta = \frac{1}{\tan \theta};$$

or

$$\csc \theta \cdot \sin \theta = 1, \quad \sec \theta \cdot \cos \theta = 1, \quad \text{ctn } \theta \cdot \tan \theta = 1.$$

To the above functions may be added versed sine (written versin), the coversed sine (written coversin), and the external secant (written exsec), which are defined by the equations versin $\theta = 1 - \cos\theta$, coversin $\theta = 1 - \sin\theta$, and exsec $\theta = \sec\theta - 1$.

It is left as an exercise to trace the variation of $\csc\theta$, $\sec\theta$, $\operatorname{ctn}\theta$, as θ varies from 0° to 360°. Be careful to note that ctn 0°, ctn 180°, csc 0°, csc 180°, sec 90°, sec 270° are undefined. Why?

116. The Representation of the Functions by Lines. We have seen in §§ 109–111, that if we take a unit circle we may represent $\sin\theta$, $\cos\theta$, and $\tan\theta$ by means of lines. We will now extend this representation to include $\csc\theta$, $\sec\theta$, $\operatorname{ctn}\theta$.

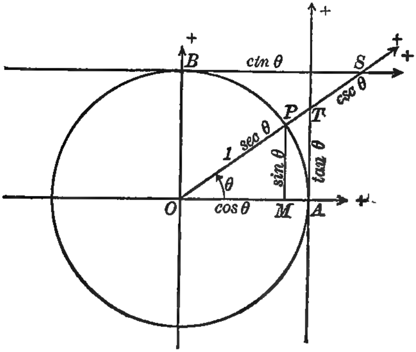


FIG. 107

Figure 107 shows the functions in a unit circle for an angle θ in the first quadrant. We have

$$MP = \sin\theta \qquad AT = \tan\theta \qquad OT = \sec\theta$$
$$OM = \cos\theta \qquad BS = \operatorname{ctn}\theta \qquad OS = \csc\theta$$

Draw similar figures for angles in each of the other quadrants. The points may be so labeled that the results given for the first quadrant hold in any quadrant.

117. Relations among the Trigonometric Functions. As one might imagine, the six trigonometric functions sine, cosine, tangent, cosecant, secant, cotangent are connected by certain relations. We shall now find some of these relations.

From Fig. 80 (§ 102) it is seen that for all cases we have

$$(1) \qquad x^2 + y^2 = r^2.$$

If we divide both sides of (1) by r^2, we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \quad \text{(by hypothesis } r \neq 0\text{)};$$

or

$$\mathbf{\sin^2 \theta + \cos^2 \theta = 1.}$$

Dividing both sides by x^2, we have

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2} \quad \text{(if } x \neq 0\text{)}.$$

Therefore

$$\mathbf{1 + \tan^2 \theta = \sec^2 \theta.}$$

Similarly dividing both sides of (1) by y^2 gives

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2} \quad \text{(if } y \neq 0\text{)};$$

or

$$\mathbf{\text{ctn}^2 \theta + 1 = \csc^2 \theta.}$$

Moreover, we have

$$\tan \theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin \theta}{\cos \theta}$$

and, similarly,

$$\text{ctn}\, \theta = \frac{\cos \theta}{\sin \theta}.$$

118. Identities. By means of the relations just proved any expression containing trigonometric functions may be put into a number of different forms. It is often of the greatest importance to notice that two expressions, although of a different form, are nevertheless identical in value. (See § 47 for the definition of an identity.)

The truth of an identity is usually established by reducing both sides, either to the same expression, or to two expressions which we know to be identical. The following examples will illustrate the methods used.

EXAMPLE 1. Prove the relation $\sec^2\theta + \csc^2\theta = \sec^2\theta\csc^2\theta$.

We may write the given equation in the form

$$\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \sec^2\theta\csc^2\theta,$$

or

$$\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} = \sec^2\theta\csc^2\theta,$$

or

$$\frac{1}{\cos^2\theta\sin^2\theta} = \sec^2\theta\csc^2\theta,$$

which reduces to

$$\sec^2\theta\csc^2\theta = \sec^2\theta\csc^2\theta.$$

Since this is an identity, it follows, by retracing the steps, that the given equality is identically true.

Both members of the given equality are undefined for the angles 0°, 90°, 180°, 270°, 360° or any multiples of these angles.

EXAMPLE 2. Prove the identity $1 + \sin\theta = \dfrac{\cos^2\theta}{1 - \sin\theta}$.

Since $\cos^2\theta = 1 - \sin^2\theta$, we may write the given equation in the form $1 + \sin\theta = \dfrac{1 - \sin^2\theta}{1 - \sin\theta}$ or $1 + \sin\theta = 1 + \sin\theta$.

As in Example 1, this shows that the given equality is identically true.

The right-hand member has no meaning when $\sin\theta = 1$, while the left-hand member is defined for all angles. We have, therefore, proved that the two members are equal except for the angle 90° or $(4n + 1)\,90°$, where n is any integer.

The formulas of § 117 may be used to solve examples of the type given in § 107.

EXAMPLE 3. Given that $\sin\theta = \frac{5}{13}$ and that $\tan\theta$ is negative, find the values of the other trigonometric functions.

Since $\sin^2\theta + \cos^2\theta = 1$, it follows that $\cos\theta = \pm\frac{12}{13}$, but since $\tan\theta$ is negative, θ lies in the second quadrant and $\cos\theta$ must be $-\frac{12}{13}$. Moreover, the relation $\tan\theta = \sin\theta/\cos\theta$ gives $\tan\theta = -\frac{5}{12}$. The reciprocals of these functions give $\sec\theta = -\frac{13}{12}$, $\csc\theta = \frac{13}{5}$, $\operatorname{ctn}\theta = -\frac{12}{5}$.

EXERCISES

1. Define secant of an angle; cosecant; cotangent.

2. Are there any angles for which the secant is undefined? If so, what are the angles? Answer the same questions for cosecant and cotangent.

3. Define versed sine; coversed sine.

4. Complete the following formulas:

$$\sin^2\theta + \cos^2\theta = ? \qquad 1 + \tan^2\theta = ? \qquad 1 + \operatorname{ctn}^2\theta = ? \qquad \tan\theta = ?$$

Do these formulas hold for all angles?

5. In what quadrants is the secant positive? negative? the cosecant positive? negative? cotangent positive? negative?

6. Is there an angle whose tangent is positive and whose cotangent is negative?

7. In what quadrant is an angle situated if we know that

(*a*) its sine is positive and its cotangent is negative?
(*b*) its tangent is negative and its secant is positive?
(*c*) its cotangent is positive and its cosecant is negative?

8. Express $\sin^2\theta + \cos\theta$ so that it shall contain no trigonometric function except $\cos\theta$.

9. Transform $(1 + \operatorname{ctn}^2\theta)\csc\theta$ so that it shall contain only $\sin\theta$.

10. Which of the trigonometric functions are never less than one in absolute value?

11. For what angles is the following equation true: $\tan\theta = \operatorname{ctn}\theta$?

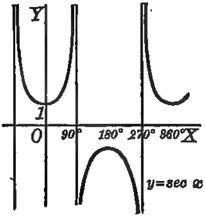
12. How many degrees are there in θ when $\operatorname{ctn}\theta = 1$? $\operatorname{ctn}\theta = -1$? $\sec\theta = \sqrt{2}$? $\csc\theta = \sqrt{2}$?

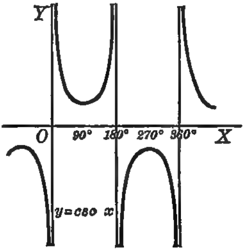
13. Determine from a figure the values of the secant, cosecant, and cotangent of 30°, 150°, 210°, 330°.

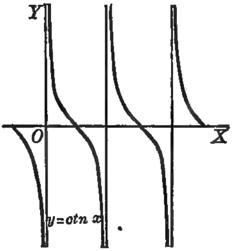
14. Determine from a figure the values of the secant, cosecant, and cotangent of 45°, 135°, 225°, 315°.

15. Determine from a figure the values of the sine, cosine, tangent, secant, cosecant, and cotangent of 60°, 120°, 240°, 300°.

16. Show that the graphs of the function $\sec\theta$, $\csc\theta$, $\operatorname{ctn}\theta$ have the forms indicated in the adjacent figures.







Prove the following identities and state for each the exceptional values of the variables, if any, for which one or both members are undefined :

17. $\cos\theta\tan\theta = \sin\theta.$

18. $\sin\theta\operatorname{ctn}\theta = \cos\theta.$

19. $\dfrac{1+\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1-\sin\theta}.$

20. $\sin^2\theta - \cos^2\theta = 2\sin^2\theta - 1.$

21. $(1-\sin^2\theta)\csc^2\theta = \operatorname{ctn}^2\theta.$

22. $\tan\theta + \operatorname{ctn}\theta = \sec\theta\csc\theta.$

23. $[x\sin\theta + y\cos\theta]^2 + [x\cos\theta - y\sin\theta]^2 = x^2 + y^2.$

24. $\dfrac{\csc\theta}{\tan\theta + \operatorname{ctn}\theta} = \cos\theta.$

25. $1 - \operatorname{ctn}^4\theta = 2\csc^2\theta - \csc^4\theta.$

26. $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta.$

27. $2(1+\sin\theta)(1+\cos\theta) = (1+\sin\theta+\cos\theta)^2.$

28. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta.$

29. $\dfrac{\csc\theta}{\csc\theta - 1} + \dfrac{\csc\theta}{\csc\theta + 1} = 2\sec^2\theta.$

30. $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\operatorname{ctn}\theta - 1}{\operatorname{ctn}\theta + 1}.$

31. $[1 + \tan\theta + \sec\theta][1 + \operatorname{ctn}\theta - \csc\theta] = 2.$

32. $(\tan\theta + \sec\theta)^2 = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$

33. $\csc^4\theta\,(1 - \cos^4\theta) - 2\operatorname{ctn}^2\theta = 1.$

34. $(\tan\theta - \operatorname{ctn}\theta)\sin\theta\cos\theta = 1 - 2\cos^2\theta.$

35. $\dfrac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta} = 1 - 2\sec\theta\tan\theta + 2\tan^2\theta.$

36. $\dfrac{\tan\alpha + \tan\beta}{\operatorname{ctn}\alpha + \operatorname{ctn}\beta} = \tan\alpha\tan\beta.$

37. $\sin\theta\,(\sec\theta + \csc\theta) - \cos\theta\,(\sec\theta - \csc\theta) = \sec\theta\csc\theta.$

Find algebraically the other trigonometric functions of the angle θ when

38. $\operatorname{ctn}\theta = 4$ and $\sin\theta$ is negative.

39. $\sin\theta = \frac{3}{5}$ and $\sec\theta$ is positive.

40. $\sec\theta = 2$ and $\tan\theta$ is negative.

41. $\csc\theta = -5$ and $\operatorname{ctn}\theta$ is positive.

119. Trigonometric Equations. An identity, as we have seen (§ 47), is an equality between two expressions which is satisfied for all values of the variables for which both expressions are defined. If the equality is not satisfied for all values of the variables for which each side is defined, it is called a conditional equality, or simply an equation. Thus $1 - \cos\theta = 0$ is true only if $\theta = n \cdot 360°$, where n is an integer. To solve a trigonometric equation, *i.e.* to find the values of θ for which the equality is true, we usually proceed as follows.

1. Express all the trigonometric functions involved in terms of one trigonometric function of the *same* angle.

2. Find the value (or values) of this function by ordinary algebraic methods.

3. Find the angles between 0° and 360° which correspond to the values found. These angles are called *particular solutions.*

4. Give the general solution by adding $n \cdot 360°$, where n is any integer, to the particular solutions.

EXAMPLE 1. Find θ when $\sin\theta = \frac{1}{2}$.

The particular solutions are 30° and 150°. The general solutions are $30° + n \cdot 360°$, $150° + n \cdot 360°$.

EXAMPLE 2. Solve the equation $\tan\theta\sin\theta - \sin\theta = 0$.

Factoring the expression, we have $\sin\theta\,(\tan\theta - 1) = 0$. Hence we have $\sin\theta = 0$, or $\tan\theta - 1 = 0$. Why?

The particular solutions are therefore 0°, 180°, 45°, 225°. The general solutions are $n \cdot 360°$, $180° + n \cdot 360°$, $45° + n \cdot 360°$, $225° + n . 360°$.

EXAMPLE 3. Find θ when $\tan\theta + \text{ctn}\,\theta = 2$.

The given equation may be written

$$\tan\theta + \frac{1}{\tan\theta} = 2,$$

or

$$\tan^2\theta - 2\tan\theta + 1 = 0;$$

therefore

$$(\tan\theta - 1)^2 = 0, \quad \text{or} \quad \tan\theta = 1.$$

It follows that $\theta = 45°$ or $225°$; or, in general,

$$\theta = 45° + n \cdot 360° \text{ or } 225° + n \cdot 360°.$$

EXERCISES

Give the particular and the general solutions of the following equations:

1. $\sin\theta = \frac{1}{2}$.
2. $\sin\theta = -\frac{1}{2}$.
3. $\cos\theta = \frac{1}{2}$.
4. $\cos\theta = -\frac{1}{2}$.
5. $\sec\theta = 2$.
6. $\sec\theta = -2$.
7. $\csc\theta = 2$.
8. $\csc\theta = -2$.
9. $\tan\theta = -1$.
10. $\text{ctn}\,\theta = -1$.
11. $\tan\theta = 1$.
12. $\text{ctn}\,\theta = 1$.
13. $\tan^2\theta = 3$.
14. $\sin\theta = 0$.
15. $\cos\theta = 0$.
16. $\tan\theta = 0$.

Solve the following equations giving the particular and the general solutions in each case:

17. $\sin\theta = \cos\theta$. *Ans.* 45°, 225°; $45° + n \cdot 360°$, $225° + n \cdot 360°$.
18. $\tan^2\theta + 2\sec^2\theta = 5$.
19. $5\sin\theta + 2\cos^2\theta = 5$. *Ans.* 90°; $90° + n \cdot 360°$.
20. $\cos^2\theta + 5\sin\theta = 7$.

21. $4 \sin \theta - 3 \csc \theta = 0.$

22. $2 \sin \theta \cos^2 \theta = \sin \theta.$

23. $\cos \theta + \sec \theta = \frac{5}{2}.$

24. $2 \sin \theta = \tan \theta.$ *Ans.* Particular solutions: 0°, 180°, 60°, 300°.

25. $3 \sin \theta + 2 \cos \theta = 2.$

26. $2 \cos^2 \theta - 1 = 1 - \sin^2 \theta.$

120. The Trigonometric Functions of $-\theta$. Draw the angles θ and $-\theta$, where OP is the terminal line of θ and OP' is the terminal line of $-\theta$. Figure 108 shows an angle θ in each of the four quadrants. We shall choose $OP = OP'$ and (x, y) as the coördinates of P and (x', y') as the coördinates of P'. In all four figures

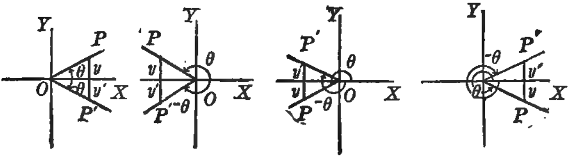


FIG. 108

$$x' = x, \; y' = -y, \; r' = r.$$

Hence

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\sin \theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos \theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\tan \theta.$$

Also,

$$\csc(-\theta) = -\csc \theta; \; \sec(-\theta) = \sec \theta; \; \text{ctn}(-\theta) = -\text{ctn}\, \theta.$$

121. The Trigonometric Functions of 90° − θ. Figure 109 represents angles θ and $90° - \theta$, when θ is in each of the four quadrants. Let OP be the terminal line of θ and OP' the terminal line of $90° - \theta$. Take $OP' = OP$ and let (x, y) be the coördinates of P and (x', y') the coördinate of P'. Then in all four figures we have

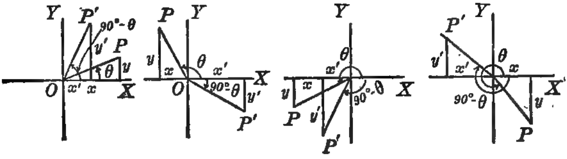


FIG. 109

$$x' = y,\ y' = x,\ r' = r.$$

Hence

$$\sin(90° - \theta) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\cos(90° - \theta) = \frac{x'}{r} = \frac{y}{r} = \sin\theta,$$

$$\tan(90° - \theta) = \frac{y'}{x'} = \frac{x}{y} = \operatorname{ctn}\theta.$$

Also,

$$\csc(90° - \theta) = \sec\theta,$$
$$\sec(90° - \theta) = \csc\theta;$$
$$\operatorname{ctn}(90° - \theta) = \tan\theta.$$

DEFINITION. The sine and cosine, the tangent and cotangent, the secant and cosecant, are called *co-functions* of each other.

The above results may be stated as follows: *Any function of an angle is equal to the corresponding co-function of the complementary angle.**

---

* Two angles are said to be *complementary* if their sum is 90°, regardless of the size of the angles.

122. The Trigonometric Functions of $180° - \theta$. By drawing figures as in §§ 120, 121, the following relations may be proved:

$$\begin{aligned} \sin(180° - \theta) &= \sin\theta, & \csc(180° - \theta) &= \csc\theta,\\ \cos(180° - \theta) &= -\cos\theta, & \sec(180° - \theta) &= -\sec\theta,\\ \tan(180° - \theta) &= -\tan\theta, & \text{ctn}(180° - \theta) &= -\text{ctn}\,\theta. \end{aligned}$$

The proof is left as an exercise.

123. The Trigonometric Functions of $180° + \theta$. Similarly, the following relations hold:

$$\begin{aligned} \sin(180° + \theta) &= -\sin\theta, & \csc(180° + \theta) &= -\csc\theta,\\ \cos(180° + \theta) &= -\cos\theta, & \sec(180° + \theta) &= -\sec\theta,\\ \tan(180° + \theta) &= \tan\theta, & \text{ctn}(180° + \theta) &= \text{ctn}\,\theta. \end{aligned}$$

The proof is left as an exercise.

124. Summary. An inspection of the results of §§ 120–123 shows:

1. *Each function of* $-\theta$ *or* $180° \pm \theta$ *is equal in absolute value* (but not always in sign) *to the same function of* θ.

2. *Each function of* $90° - \theta$ *is equal in magnitude and in sign to the corresponding co-function of* θ.

These principles enable us to find the value of any function of any angle in terms of a function of a positive acute angle (not greater than 45° if desired) as the following examples show.

EXAMPLE 1. Reduce cos 200° to a function of an angle less than 45°.

Since 200° is in the second quadrant, cos 200° is negative. Hence $\cos 200° = -\cos 20°$. Why?

EXAMPLE 2. Reduce tan 260° to a function of an angle less than 45°.

Since 260° is in the third quadrant, tan 260° is positive. Hence $\tan 260° = \tan 80° = \text{ctn}\,10°$ (§ 121).

EXERCISES

Reduce to a function of an angle not greater than 45°:

1. $\sin 163^\circ$.

2. $\cos(-110^\circ)$.
Ans. $-\cos 70^\circ$ or $-\sin 20^\circ$

3. $\sec(-265^\circ)$.

4. $\tan 428^\circ$.

5. $\csc 900^\circ$.

6. $\text{ctn}(-1215^\circ)$.

7. $\tan 840^\circ$.

8. $\sin 510^\circ$.

Find without the use of tables the values of the following functions:

9. $\cos 570^\circ$.

10. $\sin 330^\circ$.

11. $\tan 390^\circ$.

12. $\sin 420^\circ$.

13. $\cos 150^\circ$.

14. $\tan 300^\circ$.

Reduce the following to functions of positive acute angles:

15. $\sin 250^\circ$.
Ans. $-\sin 70^\circ$ or $-\cos 20^\circ$.

16. $\cos 158^\circ$.

17. $\tan(-389^\circ)$.

18. $\sec(-245^\circ)$.

19. $\csc(-321^\circ)$.

20. $\sin 269^\circ$.

21. Prove the following relations from a figure:

(*a*) $\sin(90^\circ + \theta) = \cos\theta$.
$\cos(90^\circ + \theta) = -\sin\theta$.
$\tan(90^\circ + \theta) = -\text{ctn}\,\theta$.
$\csc(90^\circ + \theta) = \sec\theta$.
$\sec(90^\circ + \theta) = -\csc\theta$.
$\text{ctn}(90^\circ + \theta) = -\tan\theta$.

(*b*) $\sin(180^\circ - \theta) = \sin\theta$.
$\cos(180^\circ - \theta) = -\cos\theta$.
$\tan(180^\circ - \theta) = -\tan\theta$.
$\csc(180^\circ - \theta) = \csc\theta$.
$\sec(180^\circ - \theta) = -\sec\theta$.
$\text{ctn}(180^\circ - \theta) = -\text{ctn}\,\theta$.

(*c*) $\sin(180^\circ + \theta) = -\sin\theta$.
$\cos(180^\circ + \theta) = -\cos\theta$.
$\tan(180^\circ + \theta) = \tan\theta$.
$\csc(180^\circ + \theta) = -\csc\theta$.
$\sec(180^\circ + \theta) = -\sec\theta$.
$\text{ctn}(180^\circ + \theta) = \text{ctn}\,\theta$.

(*d*) $\sin(270^\circ - \theta) = -\cos\theta$.
$\cos(270^\circ - \theta) = -\sin\theta$.
$\tan(270^\circ - \theta) = \text{ctn}\,\theta$.
$\csc(270^\circ - \theta) = -\sec\theta$.
$\sec(270^\circ - \theta) = -\csc\theta$.
$\text{ctn}(270^\circ - \theta) = \tan\theta$.

(*e*) $\sin(270^\circ + \theta) = -\cos\theta$.
$\cos(270^\circ + \theta) = \sin\theta$.
$\tan(270^\circ + \theta) = -\text{ctn}\,\theta$.
$\csc(270^\circ + \theta) = -\sec\theta$.
$\sec(270^\circ + \theta) = \csc\theta$.
$\text{ctn}(270^\circ + \theta) = -\tan\theta$.

125. Law of Sines. Consider any triangle ABC with the altitude CD drawn from the vertex C (Fig. 110).

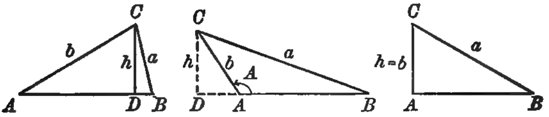


FIG. 110

In all cases we have $\sin A = \frac{h}{b}$, $\sin B = \frac{h}{a}\cdot$

Therefore, dividing, we obtain

$$\frac{\sin A}{\sin B} = \frac{a}{b},$$

or

$$\frac{a}{\sin A} = \frac{b}{\sin B}. \tag{2}$$

If the perpendicular were dropped from B, the same argument would give

$$\frac{a}{\sin A} = \frac{c}{\sin C}. \tag{3}$$

Combining results (2) and (3) we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

This law is known as the ***law of sines*** and may be stated as follows:

Any two sides of a triangle are proportional to the sines of the angles opposite these sides.

126. Law of Cosines. Consider any triangle ABC with the altitude CD drawn from the vertex C (Fig. 111).

In Fig. 111 *a*

$$AD = b \cos A;\ CD = b \sin A;\ DB = c - b \cos A.$$

In Fig. 111 *b*

$$AD = -b \cos A;\ CD = b \sin A;\ DB = c - b \cos A.$$

In both figures

$$a^2 = \overline{DB}^2 + \overline{CD}^2.$$

Therefore

$$\begin{aligned} a^2 &= c^2 - 2\,bc\cos A + b^2\cos^2 A + b^2\sin^2 A \\ &= c^2 - 2\,bc\cos A + (\cos^2 A + \sin^2 A)\,b^2, \end{aligned}$$

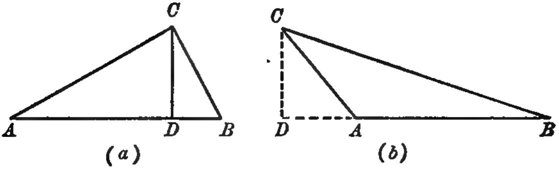


FIG. 111

whence

$$\boldsymbol{a^2 = b^2 + c^2 - 2\,bc\cos A.}$$

Similarly it may be shown that

$$\begin{aligned} b^2 &= c^2 + a^2 - 2\,ca \cdot \cos B, \\ c^2 &= a^2 + b^2 - 2\,ab \cdot \cos C. \end{aligned}$$

Any one of these similar results is called the ***law of cosines*** It may be stated as follows:

*The square of any side of a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of these two sides times the cosine of their included angle.**

127. Solution of Triangles. To solve a triangle is to find the parts not given, when certain parts are given. From geometry we know that a triangle is in general determined when three parts of the triangle, one of which is a side, are given.* Right triangles have already been solved (§ 106 ff.), and we shall now make use of the laws of sines and cosines to solve oblique triangles. The methods employed will be illustrated by some examples. It will be found advantageous to construct the triangle to scale, for by so doing one can often detect errors which may have been made.

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* Of what three theorems in elementary geometry is this the equivalent?

128. Illustrative Examples.

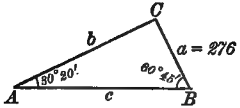


FIG. 112

EXAMPLE 1. Solve the triangle ABC, given $A = 30° 20'$, $B = 60° 45'$, $a = 276$.

SOLUTION:

$$C = 180° - (A + B) = 180° - 91° 5' = 88° 55';$$

$$b = \frac{a \sin B}{\sin A} = \frac{276 \sin 60° 45'}{\sin 30° 20'} = \frac{(276)(0.8725)}{0.5050} = 476.9;$$

also

$$c = \frac{a \sin C}{\sin A} = \frac{276 \sin 88° 55'}{\sin 30° 20'} = \frac{(276)(0.9998)}{0.5050} = 546.4.$$

CHECK: It is left as an exercise to show that for these values we have

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

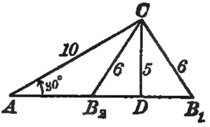


FIG. 113

EXAMPLE 2. Solve the triangle ABC, given $A = 30°$, $b = 10$, $a = 6$.

Constructing the triangle ABC, we see that two triangles AB_1C and AB_2C answer the description since $b > a >$ altitude CD.

SOLUTION: Now

$$\frac{\sin B_1}{\sin A} = \frac{b}{a}, \text{ or } \sin B_1 = \frac{b \sin A}{a} = 0.833,$$

whence $B_1 = 56.5°$.

But

$$B_2 = 180° - B_1 = 180° - 56.5° = 123.5°,$$

and

$$C_1 = 180° - (A + B_1) = 180° - 86.5° = 93.5°,$$
$$C_2 = 180° - (A + B_2) = 180° - 153.5° = 26.5°.$$

---

* When two sides and an angle opposite one of them are given, the triangle is not always determined. Why?

Now
$$\frac{c_2}{a}=\frac{\sin C_2}{\sin A},\quad \text{or}\quad c_2=\frac{a\sin C_2}{\sin A}=\frac{(6)(0.446)}{0.500}=5.35.$$
Also
$$\frac{c_1}{a}=\frac{\sin C_1}{\sin A},\quad \text{or}\quad c_1=\frac{a\sin C_1}{\sin A}=\frac{(6)(0.998)}{0.500}=11.98.$$

CHECK:
$$c_1^2=a^2+b^2-2\,ab\cos C_1.$$
$$143.5=36+100+(2)(6)(10)(0.061)=143.3.$$
$$c_2^2=a^2+b^2-2\,ab\cos C_2.$$
$$28.62=36+100-(2)(6)(10)(0.895)=28.60.$$

EXAMPLE 3. Solve the triangle ABC, given $a=10$, $b=6$, $C=40°$.

SOLUTION: $c^2=a^2+b^2-2\,ab\cos C$
$$=100+36-(120)(0.766)=44.08.$$
Therefore $c=6.64$. Now
$$\sin A=\frac{a\sin C}{c}=\frac{(10)(0.643)}{6.64}=0.968,$$
i.e. $A=104.5°$. Likewise
$$\sin B=\frac{b\sin C}{c}=\frac{(6)(0.643)}{6.64}=0.581,$$
i.e. $B=35.5°$.

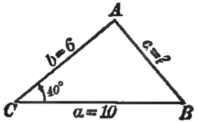


FIG. 114

CHECK: $A+B+C=180.0°$.

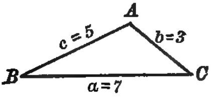


FIG. 115

EXAMPLE 4. Solve the triangle ABC when $a=7$, $b=3$, $c=5$.

From the law of cosines,
$$\cos A=\frac{b^2+c^2-a^2}{2\,bc}=-\frac{1}{2}=-0.500,$$
$$\cos B=\frac{a^2+c^2-b^2}{2\,ac}=\frac{13}{14}=0.928,$$
$$\cos C=\frac{a^2+b^2-c^2}{2\,ab}=\frac{11}{14}=0.786.$$
Therefore
$$A=120°,\ B=21.8°,\ C=38.2°.$$
CHECK: $A+B+C=180.0°$.

EXERCISES

1. Solve the triangle ABC, given

(*a*) $A=30°$, $B=70°$, $a=100$;
(*b*) $A=40°$, $B=70°$, $c=110$;
(*c*) $A=45.5°$, $C=68.5°$, $b=40$;
(*d*) $B=60.5°$, $C=44°\,20'$, $c=20$;

(*e*) $a = 30$, $b = 54$, $C = 50°$;
(*f*) $b = 8$, $a = 10$, $C = 60°$;
(*g*) $a = 10$, $b = 12$, $c = 14$;
(*h*) $a = 21$, $b = 24$, $c = 28$.

2. Determine the number of solutions of the triangle ABC when

(*a*) $A = 30°$, $b = 100$, $a = 70$;
(*b*) $A = 30°$, $b = 100$, $a = 100$;
(*c*) $A = 30°$, $b = 100$, $a = 50$;
(*d*) $A = 30°$, $b = 100$, $a = 40$;
(*e*) $A = 30°$, $b = 100$, $a = 120$;
(*f*) $A = 106°$, $b = 120$, $a = 16$;
(*g*) $A = 90°$, $b = 15$, $a = 14$.

3. Solve the triangle ABC when

(*a*) $A = 37° 20'$, $a = 20$, $b = 26$;
(*b*) $A = 37° 20'$, $a = 40$, $b = 26$;
(*c*) $A = 30°$, $a = 22$, $b = 34$.

4. In order to find the distance from a point A to a point B, a line AC and the angles CAB and ACB were measured and found to be 300 yd., 60° 30′, 56° 10′ respectively. Find the distance AB.

5. In a parallelogram one side is 40 and one diagonal 90. The angle between the diagonals (opposite the side 40) is 25°. Find the length of the other diagonal and the other side. How many solutions?

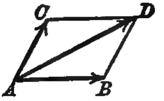
6. Two observers 4 miles apart, facing each other, find that the angles of elevation of a balloon in the same vertical plane with themselves are 60° and 40° respectively. Find the distance from the balloon to each observer and the height of the balloon.

7. Two stakes A and B are on opposite sides of a stream; a third stake C is set 100 feet from A, and the angles ACB and CAB are observed to be 40° and 110°, respectively. How far is it from A to B?

8. The angle between the directions of two forces is 60°. One force is 10 pounds and the resultant of the two forces is 15 pounds. Find the other force.[*]

9. Resolve a force of 90 pounds into two equal components whose directions make an angle of 60° with each other.

10. An object B is wholly inaccessible and invisible from a certain point A. However, two points C and D on a line with A may be found such that from these points B is visible. If it is found that $CD =$ 300 feet, $CA = 120$ feet, angle $DCB = 70°$, angle $CDB = 50°$, find the length AB.



[*] It is shown in physics that if the line segments AB and AC represent in magnitude and direction two forces acting at a point A, then the diagonal AD of the parallelogram $ABCD$ represents both in magnitude and direction the resultant of the two given forces.

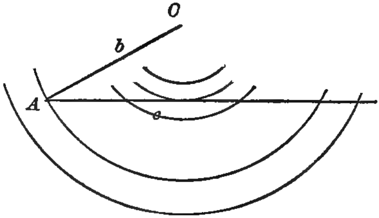
11. Given a, b, A, in the triangle ABC. Show that the number of possible solutions are as follows:

$A < 90°$

$$\begin{cases} a < b \sin A & \text{no solution,} \\ b \sin A < a < b & \text{two solutions,} \\ \left.\begin{array}{l} a \geqq b \\ a = b \sin A \end{array}\right\} & \text{one solution.} \end{cases}$$

$A \geqq 90°$

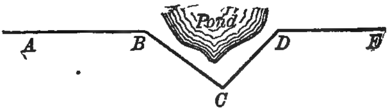
$$\begin{cases} a \leqq b & \text{no solution,} \\ a > b & \text{one solution.} \end{cases}$$



12. The diagonals of a parallelogram are 14 and 16 and form an angle of 50°. Find the length of the sides.

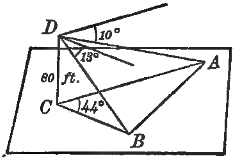
13. Resolve a force of magnitude 150 into two components of 100 and 80 and find the angle between these components.

14. It is sometimes desirable in surveying to extend a line such as AB in the adjoining figure. Show that this can be done by means of the broken line $ABCDE$. What measurements are necessary?



15. Three circles of radii 2, 6, 5 are mutually tangent. Find the angles between their lines of centers.

16. In order to find the distance between two objects A and B on opposite sides of a house, a station C was chosen, and the distances $CA = 500$ ft., $CB = 200$ ft., together with the angle $ACB = 65° \, 30'$ were measured. Find the distance from A to B.



17. The sides of a field are 10, 8, and 12 rods respectively. Find the angle opposite the longer side.

18. From a tower 80 feet high, two objects, A and B, in the plane of the base are found to have angles of depression of 13° and 10° respectively; the horizontal angle subtended by A and B at the foot C of the tower is 44°. Find the distance from A to B.

129. Areas of Oblique Triangles.

1. *When two sides and the included angle are given.*

Denoting the area by S, we have from geometry

$$S = \tfrac{1}{2}\, ch,$$

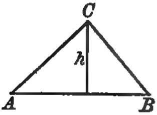


FIG. 116

but $h = b \sin A$; therefore

$$S = \tfrac{1}{2}\, cb \sin A. \tag{4}$$

Likewise,

$$S = \tfrac{1}{2}\, ab \sin C \quad \text{and} \quad S = \tfrac{1}{2}\, ac \sin B.$$

2. *When a side and two adjacent angles are given.*

Suppose the side a and the adjacent angles B and C to be given. We have just seen that $S = \frac{1}{2}\, ac \sin B$. But from the law of sines we have

$$c = \frac{a \sin C}{\sin A}.$$

Therefore

$$S = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \sin A}.$$

But $\sin A = \sin [180° - (B + C)] = \sin (B + C)$. Therefore

$$S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)}.$$

3. *When the three sides are given.*

We have seen that $S = \frac{1}{2}\, bc \sin A$. Squaring both sides of this formula and transforming, we have

$$S^2 = \frac{b^2c^2}{4} \sin^2 A = \frac{b^2c^2}{4}(1 - \cos^2 A)$$

$$= \frac{bc}{2}(1 + \cos A) \cdot \frac{bc}{2}(1 - \cos A);$$

whence,

$$S^2 = \frac{bc}{2}\left(1 + \frac{b^2 + c^2 - a^2}{2\, bc}\right) \cdot \frac{bc}{2}\left(1 - \frac{b^2 + c^2 - a^2}{2\, bc}\right)$$

$$= \frac{2\,bc + b^2 + c^2 - a^2}{4} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{4}$$

$$= \frac{b+c+a}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2},$$

which may be written in the form

$$S^2 = s(s-a)(s-b)(s-c),$$

where $2\,s = a + b + c$. Therefore,

$$S = \sqrt{s(s-a)(s-b)(s-c)}. \tag{5}$$

130. The Radius of the Inscribed Circle. If r is the radius of the inscribed circle, we have from elementary geometry, since s is half the perimeter of the triangle, $S = rs$; equating this value of S to that found in equation (5) of the last article and then solving for r, we get,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

EXERCISES

Find the area of the triangle ABC, given

1. $a = 25,\ b = 31.4,\ C = 80°\ 25'$.

2. $b = 24,\ c = 34\ 3,\ A = 60°\ 25'$.

3. $a = 37,\ b = 13,\ C = 40°$.

4. $a = 10,\ b = 7,\ C = 60°$.

5. $a = 10,\ b = 12,\ C = 60°$.

6. $a = 10,\ b = 12,\ C = 8°$.

7. Find the area of a parallelogram in terms of two adjacent sides and the included angle.

8. The base of an isosceles triangle is 20 ft. and the area is $100/\sqrt{3}$ sq. ft. Find the angles of the triangle. *Ans.* 30°, 30°, 120°.

9. Find the radius of the inscribed circle of the triangle whose sides are 12, 10, 8.

10. How many acres are there in a triangular field having one of its sides 50 rods in length and the two adjacent angles, respectively, 70° and 60°?

CHAPTER VII

TRIGONOMETRIC RELATIONS

131. Radian Measure. In certain kinds of work it is more convenient in measuring angles to use, instead of the degree, a unit called the radian. A ***radian*** is defined as the angle at the center of a circle whose subtended arc is equal in length to the radius of the circle (Fig. 117). Therefore, if an angle θ at the center of a circle of radius r units subtends an arc of s units, the measure of θ in radians is

$$(1) \qquad \theta = \frac{s}{r}.$$

Since the length of the whole circle is $2\pi r$, it follows that

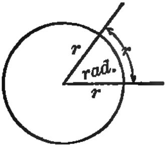


FIG. 117

$$\frac{2\pi r}{r} = 2\pi \text{ radians} = 360°,$$

or

$$(2) \qquad \boldsymbol{\pi \text{ radians} = 180°.}$$

Therefore,

$$1 \text{ radian} = \frac{180°}{\pi} = 57°\,17'\,45'' \text{ (approximately).}$$

It is important to note that the radian* as defined is a constant angle, *i.e.*, it is the same for all circles, and can therefore be used as a unit of measure.

---

[*] The symbol ^{r} is often used to denote radians. Thus 2^r stands for 2 radians, π^r for π radians, etc. When the angle is expressed in terms of π (the radian being the unit), it is customary to omit ^{r}. Thus, when we refer to an angle π, we mean an angle of π radians. When the word radian is omitted, it should be mentally supplied in order to avoid the error of supposing π means 180. Here, as in geometry, $\pi = 3.14159. . . .$

From relation (2) it follows that to convert radians into degrees it is only necessary to multiply the number of radians by $180/\pi$, while to convert degrees into radians we multiply the number of degrees by $\pi/180$. Thus 45° is $\pi/4$ radians; $\pi/2$ radians is 90°.

132. The Length of Arc of a Circle. From relation (1), § 131, it follows that

$$s = r\theta.$$

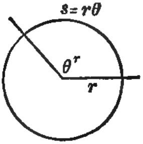


FIG. 118

That is (Fig. 118), if a central angle is measured in radians, and if its intercepted arc and the radius of the circle are measured in terms of the *same unit*, then

length of arc = radius × central angle in radians.

EXERCISES

1. Express the following angles in radians:

25°, 145°, 225°, 300°, 270°, 450°, 1150°.

2. Express in degrees the following angles:

$$\frac{\pi}{4}, \; -\frac{7\pi}{6}, \; \frac{5\pi}{6}, \; 3\pi, \; \frac{5\pi}{4}.$$

3. A circle has a radius of 20 inches. How many radians are there in an angle at the center subtended by an arc of 25 inches? How many degrees are there in this same angle? *Ans.* $\frac{5}{4}^r$; 71° 37′ approx.

4. Find the radius of a circle in which an arc 12 inches long subtends an angle of 35°.

5. The minute hand of a clock is 4 feet long. How far does its extremity move in 22 minutes?

6. In how many hours is a point on the equator carried by the rotation of the earth on its axis through a distance equal to the diameter of the earth?

7. A train is traveling at the rate of 10 miles per hour on a curve of half a mile radius. Through what angle has it turned in one minute?

8. A wheel 10 inches in diameter is belted to a wheel 3 inches in diameter. If the first wheel rotates at the rate of 5 revolutions per minute, at what rate is the second rotating? How fast must the former rotate in order to produce 6000 revolutions per minute in the latter?

133. Angular Measurement in Artillery Service. The divided circles by means of which the guns of the United States Field Artillery are aimed are graduated neither in degrees nor in radians, but in units called *mils.* The mil is defined as an angle subtended by an arc of $\frac{1}{6400}$ of the circumference, and is therefore equal to

$$\frac{2\pi}{6400} = \frac{3.1416}{3200} = 0.00098175 = (0.001 - 0.00001825) \text{ radian.}$$

The mil is therefore approximately one thousandth of a radian. (Hence its name.)*

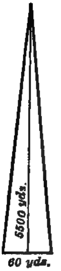


FIG. 119

Since (§ 132)

$$\text{length of arc} = \text{radius} \times \text{central angle in radians},$$

it follows that we have *approximately*

$$\text{length of arc} = \frac{\text{radius}}{1000} \times \text{central angle in mils};$$

i.e. length of arc in yards = (radius in thousands of yards) · (angle in mils). The error here is about 2 %.

EXAMPLE 1. A battery occupies a front of 60 yd. If it is at 5500 yd. range, what angle does it subtend (Fig. 119)? We have, evidently,

$$\text{angle} = \frac{60}{5.5} = 11 \text{ mils.}$$

EXAMPLE 2. *Indirect Fire.*† A battery posted with its right gun at G is to open fire on a battery at a point T, distant 2000 yd. and invisible

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[*] To give an idea of the value in mils of certain angles the following has been taken from the *Drill Regulations for Field Artillery* (1911), p. 164:

"Hold the hand vertically, palm outward, arm fully extended to the front. Then the angle subtended by the

width of thumb is	40 mils
width of first finger at second joint is	40 mils
width of second finger at second joint is	40 mils
width of third finger at second joint is	35 mils
width of little finger at second joint is	30 mils
width of first, second, and third fingers at second joint is	115 mils

These are average values."

[†] The limits of this text preclude giving more than a single illustration of the problems arising in artillery practice. For other problems the student is referred to the *Drill Regulations for Field Artillery* (1911), pp. 57, 61, 150–164; and to ANDREWS, *Fundamentals of Military Service*, pp. 153–159, from which latter text the above example is taken.

from G (Fig. 120). The officer directing the fire takes post at a point B from which both the target T and a church spire P, distant 3000 yd. from G are visible. B is 100 yd. at the right of the line GT and 120 yd. at the right of the line GP and the officer finds by measurement that the angle PBT contains 3145 mils. In order to train the gun on the target the gunner must set off the angle PGT on the sight of the piece and then move the gun until the spire P is visible through the sight. When this is effected, the gun is aimed at T.

Let F and E be the feet of the perpendiculars from B to GT and GP respectively, and let BT' and BP' be the parallels to GT and GP that pass through B. Then, evidently, if the officer at B measures the angle PBT, which would be used instead of angle PGT were the gun at B instead of at G, and determines the angles $TBT' = FTB$ and $PBP' = EPB$, he can find the angle PGT from the relation

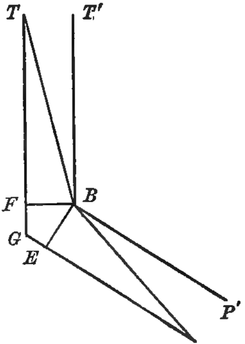

FIG. 120

$$PGT = P'BT' = PBT - TBT' - PBP'.$$

Now $$\tan FTB = \frac{FB}{TF}, \quad \tan EPB = \frac{EB}{PE}.$$

Furthermore if FTB and EPB are small angles, *i.e.*, if FB and EB are small compared with GT and GP respectively, the radian measure of the angle is approximately equal to the tangent of the angle. Why? Hence we have

$$\left.\begin{aligned} FTB &= \tan FTB = \frac{FB}{GT} \\ EPB &= \tan EPB = \frac{EB}{GP} \end{aligned}\right\} \text{approximately.}$$

Therefore $$TBT' = FTB = \frac{100}{2000} \text{ radians} = 50 \text{ mils},$$

$$PBP' = EPB = \frac{120}{3000} \text{ radians} = 40 \text{ mils}.$$

Hence $$\begin{aligned} PGT &= PBT - TBT' - PBP' \\ &= 3145 - 50 - 40 \\ &= 3055 \text{ mils}, \end{aligned}$$

which is the angle to be set off on the sight of the gun.

Hence for the situation indicated in Fig. 120 we have the following rule:*

(1) Measure in mils the angle PBT from the aiming point P to the target T as seen at B.

(2) Measure or estimate the offsets FB and EB in yards, the range GT and the distance GP of the aiming point P in *thousands* of yards.

(3) Compute in mils the offset angles by means of the relations

$$TBT' = FTB,$$
$$PBP' = EPB,$$
$$TBT' = \frac{FB}{GT},$$
$$PBP' = \frac{EB}{GP}.$$

(4) Then the angle of deflection PGT is equal to the angle PBT diminished by the sum of the offset angles.

EXERCISES

1. A battery occupies a front of 80 yd. It is at 5000 yd. range. What angle does it subtend?

2. In Fig. 120 suppose $PBT = 3000$ mils, $FB = 200$ yd., $GT = 3000$ yd., $EB = 150$ yd., $GP = 4000$ yd. Find the number of mils in PGT.

3. A battery at a point G is ordered to take a masked position and be ready to fire on an indicated hostile battery at a point T whose range is known to be 2100 yd. The battery commander finds an observing station B, 200 yd. at the right and on the prolongation of the battery front, and 175 yd. at the right of PG. An aiming point P, 5900 yd. in the rear, is found, and PBT is found to be 2600 mils. Find PGT.

134. Inverse Trigonometric Functions. The equation

$$x = \sin y \tag{1}$$

may be read:

y is an angle whose sine is equal to *x*,

a statement which is usually written in the contracted form

$$y = \text{arc} \sin x.\dagger \tag{2}$$

---

* There are three cases with corresponding rules, depending on whether P is in front of, rear of, or on the flank of G.

† Sometimes written $y = \sin^{-1} x$. Here -1 is not an algebraic exponent, but merely a part of a functional symbol. When we wish to raise $\sin x$ to the power -1, we write $(\sin x)^{-1}$.

For example, $x = \sin 30^\circ$ means that $x = \frac{1}{2}$, while $y = \text{arc}\sin\frac{1}{2}$ means that $y = 30^\circ$, 150°, or in general (n being an integer),

$$30^\circ + n \cdot 360^\circ;\ 150^\circ + n \cdot 360^\circ.$$

Since the sine is never greater than 1 and never less than -1, it follows that $-1 \leqq x \leqq 1$. It is evident that there is an unlimited number of values of $y = \text{arc}\sin x$ for a given value of x in this interval.

We shall now define the *principal value* Arc sin x* of arc sin x, distinguished from arc sin x by the use of the capital A, to be the numerically smallest angle whose sine is equal to x. This function like arc sin x is defined only for those values of x for which

$$-1 \leqq x \leqq 1.$$

The difference between arc sin x and Arc sin x is well illustrated by means of their graph. It is evident that the graph of $y = \text{arc}\sin x$, *i.e.* $x = \sin y$ is simply the sine curve with the rôle of the x and y axes interchanged. (See Fig. 121.) Then for every admissible value of x, there is an unlimited number of values of y; namely, the ordinates of all the points P_1, P_2, $\cdots$, in which a line at a distance x and parallel to the y-axis intersects the curve. The single-valued function Arc sin x is represented by the part of the graph between M and N.

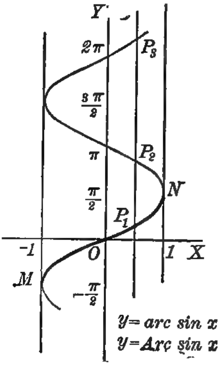

FIG. 121

Similarly arc cos x, defined as "an angle whose cosine is x,"

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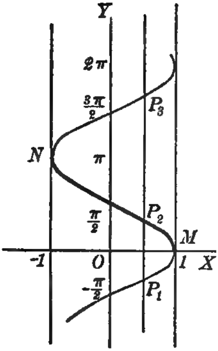
* Sometimes written $\text{Sin}^{-1}x$, distinguished from $\sin^{-1}x$ by the use of the capital S.



$y=$ arc cos x
$y=$Arc cos x
FIG. 122

has an unlimited number of values for every admissible value of x $(-1 \leqq x \leqq 1)$. We shall define the principal value Arc cos x as the smallest positive angle whose cosine is x. That is,

$$0 \leqq \text{Arc}\cos x \leqq \pi.$$

Figure 122 represents the graph of $y=\text{arc}\cos x$ and the portion of this graph between M and N represents Arc cos x.

Similarly we write $x=\tan y$ as $y=\text{arc}\tan x$, and in the same way we define the symbols arc ctn x; arc sec x; arc csc x. The principal values of all the inverse trigonometric functions are given in the following table.

$y=$	Arc sin x	Arc cos x	Arc tan x
Range of x	$-1 \leqq x \leqq 1$	$-1 \leqq x \leqq 1$	all real values
Range of y	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$	0 to π	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$
x positive	1st Quad.	1st Quad.	1st Quad.
x negative	4th Quad.	2d Quad.	4th Quad.
	Arc ctn x	Arc sec x	Arc csc x
Range of x	all values	$x \geqq 1$ or $x \leqq -1$	$x \geqq 1$ or $x \leqq -1$
Range of y	0 to π	0 to π	$-\frac{\pi}{2}$ to $\frac{\pi}{2}$
x positive	1st Quad.	1st Quad.	1st Quad.
x negative	2d Quad.	2d Quad.	4th Quad.

In so far as is possible we select the principal value of each inverse function, and its range, so that the function is single-valued, continuous, and takes on all possible values. This obviously cannot be done for the Arc sec x and for Arc csc y.

EXERCISES

1. Explain the difference between arc sin x and Arc sin x.

2. Find the values of the following expressions:

(*a*) Arc sin $\frac{1}{2}$. (*b*) arc sin $\frac{1}{2}$. (*c*) arc tan 1.
(*d*) Arc tan -1. (*e*) arc cos $\frac{\sqrt{3}}{2}$. (*f*) Arc cos $\frac{\sqrt{3}}{2}$.

3. What is meant by the angle π? $\pi/4$?

4. Through how many radians does the minute hand of a watch turn in 30 minutes? in one hour? in one and one half hours?

5. For what values of x are the following functions defined:

(*a*) arc sin x? (*b*) arc cos x? (*c*) arc tan x?
(*d*) arc ctn x? (*e*) arc sec x? (*f*) arc csc x?

6. What is the range of values of the functions:

(*a*) Arc sin x? (*b*) Arc cos x? (*c*) Arc tan x?
(*d*) Arc ctn x? (*e*) Arc sec x? (*f*) Arc csc x?

7. Draw the graph of the functions:

(*a*) arc sin x. (*b*) arc cos x. (*c*) arc tan x.
(*d*) arc ctn x. (*e*) arc sec x. (*f*) arc csc x.

8. Find the value of cos (Arc tan $\frac{3}{4}$).

HINT. Let Arc tan $\frac{3}{4} = \theta$. Then tan $\theta = \frac{3}{4}$ and we wish to find the value of cos θ.

9. Find the values of cos (arc tan $\frac{3}{4}$).

10. Find the value of the following expressions:

(*a*) sin (arc cos $\frac{3}{5}$). (*c*) cos (Arc cos $\frac{5}{13}$). (*e*) sin (Arc sin $\frac{1}{5}$).
(*b*) sin (arc sec 3). (*d*) sec (Arc csc 2). (*f*) tan (Arc tan 5).

11. Prove that Arc sin $(2/5) =$ Arc tan $(2/\sqrt{21})$.

12. Find x when Arc cos $(2x^2 - 2x) = 2\pi/3$.

Find the values of the following expressions:

13. cos [90° − Arc tan $\frac{3}{4}$].

14. sec [90° − Arc sec 2].

15. tan [90° − Arc sin $\frac{5}{13}$].

135. Projection. Consider two directed lines p and q in a plane, *i.e.* two lines on each of which one of the directions has been specified as positive (Fig. 123). Let A and B be any two points on p and let A', B' be the points in which perpendiculars to q through A and B, respectively, meet q. The directed segment $A'B'$ is called the *projection of the directed segment AB on q* and is denoted by

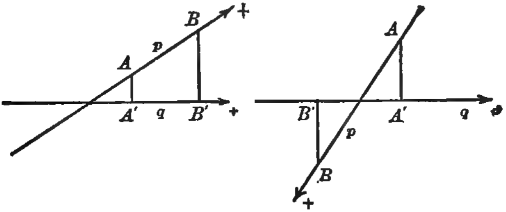


FIG. 123

$$A'B' = \text{proj}_q\, AB.$$

In both figures AB is positive. In the first figure $A'B'$ is positive, while in the second figure it is negative.

As special cases of this definition we note the following:

1. If p and q are parallel and are directed in the same way, we have

$$\text{proj}_q\, AB = AB.$$

2. If p and q are parallel and are directed oppositely, we have

$$\text{proj}_q\, AB = -AB.$$

3. If p is perpendicular to q, we have

$$\text{proj}_q\, AB = 0.$$

It should be noted carefully that these propositions are true no matter how A and B are situated on p.

We may now prove the following important proposition:

If A, B are any two points on a directed line p, and q is any directed line in the same plane with p, then we have both in magnitude and sign:

$$\text{(1)} \qquad \textit{proj}_q\, AB = AB \cdot \cos(qp),$$

where (qp) represents an angle through which q must be rotated in order to make its direction coincide with the direction of p.

We note first that all possible determinations of the angle (qp) have the same cosine, since any two of these determinations differ by multiples of 360° (Fig. 124). We shall prove the proposition first for the case where AB has the same direction as p, *i.e.* where AB is positive. To this end we draw through A (Fig. 125) a line q_1 parallel to q and directed in the same way. (We may evidently assume without loss of generality that q is horizontal and is directed to the right.)

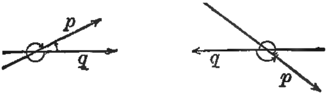


FIG. 124

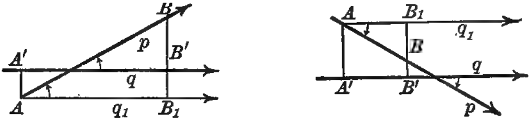


FIG. 125

Let $A'B'$ have the same significance as before and let BB' meet q_1 in B_1. Then, by the definition of the cosine, we have

$$\frac{AB_1}{AB} = \cos(q_1 p) = \cos(qp),$$

in magnitude and in sign; or

$$AB_1 = AB \cos(qp).$$

But

$$AB_1 = A'B' = \text{proj}_q\, AB.$$

Therefore

$$\text{proj}_q\, AB = AB \cos(qp).$$

Finally, if AB is negative, BA is positive, and, by the result just obtained, we should have

$$B'A' = BA \cos(qp).$$

Hence, changing signs on both sides of this equation, we have

$$A'B' = AB \cos(qp).$$

The special cases 1, 2, 3 listed on p. 196 are obtained from formula (1) by placing (qp) equal to 0°, 180°, 90°, respectively; for $\cos 0° = 1$, $\cos 180° = -1$, $\cos 90° = 0$.

136. Application of Projection. In Physics, forces and velocities are usually represented by line segments. A force of 20 pounds, for example, is represented by a segment 20 units in length and drawn in the direction of the force. A velocity of 20 feet per second is represented by a segment 20 units in length and drawn in the direction of the motion.

The projection on a given line l of a segment representing a force or velocity represents the *component* of the force or velocity in the direction of l.

EXAMPLE. A smooth block is sliding down a smooth incline which makes an angle of 30° with the horizontal. If the block weighs 10 lb., what force acting directly up the plane will keep the block at rest?

Draw the segment AB 10 units in length, directly downward to represent the force exerted by the weight. Project this segment on the incline and call this projection AC. Now angle $ABC = 30°$. Therefore $AC = AB \sin 30° = 5$. This is the component of the force AB down the plane. Therefore a force of 5 lb. acting up the plane will keep the body at rest.

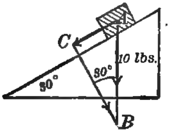


FIG. 126

Theorem. *If A, B, C are any three points in a plane, and l is any directed line in the plane, the algebraic sum of the projections of the segments AB and BC on l is equal to the projection of the segment AC on l.*

As a point traces out the path from A to B, and then from B to C (Fig. 127), the projection of the point traces out the segments from A' to B' and then from B' to C'. The net result of this motion is a motion from A' to C' which represents the projection of AC, *i.e.*

$$A'B' + B'C' = A'C'.$$

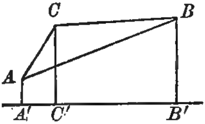


Fig. 127

EXERCISES

1. What is the projection of a line segment upon a line l, if the line segment is perpendicular to the line l?

2. Find $\text{proj}_x AB$ and $\text{proj}_y AB$* in each of the following cases, if α denotes the angle from the x-axis to AB.

(*a*) $AB = 5$, $\alpha = 60°$.
(*b*) $AB = 10$, $\alpha = 300°$.
(*c*) $AB = 6$, $\alpha = 90°$.
(*d*) $AB = 20$, $\alpha = 210°$.

3. Prove by means of projection that in a triangle ABC

$$a = b \cos C + c \cos B.$$

4. If $\text{proj}_x AB = 3$ and $\text{proj}_y AB = -4$, find the length of AB.

5. A steamer is going northeast 20 miles per hour. How fast is it going north? going east?

6. A 20 lb. block is sliding down a 15° incline. Find what force acting directly up the plane will just hold the block, allowing one half a pound for friction.

7. Prove that if the sides of a polygon are projected in order upon any given line, the sum of these projections is zero.

---

* $\text{Proj}_x AB$ and $\text{proj}_y AB$ mean the projections of AB on the x-axis and the y-axis, respectively.

137. Rotation in a Plane. Suppose that a point $P(x, y)$ in a plane moves on the arc of a circle with center at the origin O, through an angle α. Suppose that its position after this rotation is $P'(x', y')$ referred to the same axes of coördinates. We desire to find x' and y' in terms of x, y, and α.

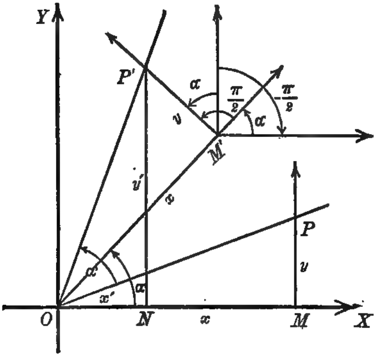


FIG. 128

In Fig. 128 we have drawn P and its coördinates $x = OM$, $y = MP$, and the new position $OM'P'$ of the triangle OMP after a rotation about the origin through an angle α. The coördinates $x' = ON$, $y' = NP'$ of P' are the projections of OP' on the x-axis and the y-axis respectively, and these projections are equal respectively to the sum of the projections of OM' and $M'P'$ on the respective axes. Hence,

$$\begin{aligned}
x' &= \text{proj}_x\, OP' = \text{proj}_x\, OM' + \text{proj}_x\, M'P' \\
&= OM' \cos(OX, OM') + M'P' \cos(OX, M'P') \\
&= x \cos\alpha + y \cos(\alpha + \pi/2) \\
&= x \cos\alpha - y \sin\alpha. \\
y' &= \text{proj}_y\, OP' = \text{proj}_y\, OM' + \text{proj}_y\, M'P' \\
&= OM' \cos(OY, OM') + M'P' \cos(OY, M'P') \\
&= x \cos(-\pi/2 + \alpha) + y \cos\alpha \\
&= x \sin\alpha + y \cos\alpha.
\end{aligned}$$

Therefore, if the point $P(x, y)$ is rotated about the origin through an angle α, the coördinates (x', y') of its new position are given by the formulas

$$(1)\qquad \begin{cases} x' = x\cos\alpha - y\sin\alpha \\ y' = x\sin\alpha + y\cos\alpha. \end{cases}$$

It should be noted that the above method of derivation is entirely general, *i.e.* it will apply to a point P in any quadrant and to any angle α.

138. The Addition Formulas. We may now enter upon a more detailed study of the properties of the trigonometric functions. We shall first express $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, $\cos\beta$.* To this end let OP be the terminal side of any angle α (Fig. 129). If OP is then rotated about O through an angle β to the position OP', the terminal line of the angle $\alpha+\beta$ is OP'. If P has the coördinates (x, y) and P' the coördinates (x', y'), then from (1) § 137,

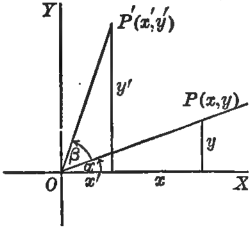


FIG. 129

$$x' = x\cos\beta - y\sin\beta,$$
$$y' = x\sin\beta + y\cos\beta.$$

Now $\sin(\alpha+\beta)$ is by definition equal to $\frac{y'}{r}$ and $\cos(\alpha+\beta)$ to $\frac{x'}{r}$ where $r = OP' = OP$. Hence

$$\sin(\alpha+\beta) = \frac{y'}{r} = \frac{x}{r}\sin\beta + \frac{y}{r}\cos\beta,$$

or

$$(1)\qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Also

$$\cos(\alpha+\beta) = \frac{x'}{r} = \frac{x}{r}\cos\beta - \frac{y}{r}\sin\beta,$$

or

$$(2)\qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

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[*] We have already had occasion to note that $\sin(\alpha+\beta)$ is *not* in general equal to $\sin\alpha + \sin\beta$. (See Ex. 5, p. 151.)

Further we have

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

Dividing numerator and denominator by $\cos\alpha\cos\beta$, we have

$$(3)\qquad \boldsymbol{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}.$$

Furthermore, by replacing β by $-\beta$ in (1), (2), and (3), and recalling that

$$\sin(-\beta)=-\sin\beta,\ \cos(-\beta)=\cos\beta,\ \tan(-\beta)=-\tan\beta,$$

we obtain

$$(4)\qquad \boldsymbol{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta},$$

$$(5)\qquad \boldsymbol{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta},$$

$$(6)\qquad \boldsymbol{\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}.$$

EXERCISES

Expand the following:

1. $\sin(45°+\alpha)=$

2. $\tan(30°-\beta)=$

3. $\cos(60°+\alpha)=$

4. $\tan(45°+60°)=$

5. $\sin(30°-45°)=$

6. $\cos(180°-45°)=$

7. What do the following formulas become if $\alpha=\beta$?

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$
$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.$$

8. Complete the following formulas:

$\sin 2\alpha\cos\alpha+\cos 2\alpha\sin\alpha=$
$\sin 3\alpha\cos\alpha-\cos 3\alpha\sin\alpha=$

$$\frac{\tan 2\alpha+\tan\alpha}{1-\tan 2\alpha\tan\alpha}=$$

9. Prove $\sin 75°=\dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $\cos 75°=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\tan 75°=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$.

10. Given $\tan\alpha=\frac{3}{4}$, $\sin\beta=\frac{5}{13}$, and α and β both positive acute angles, find the value of $\tan(\alpha+\beta)$; $\sin(\alpha-\beta)$; $\cos(\alpha+\beta)$; $\tan(\alpha-\beta)$.

11. Prove that

(*a*) $\cos(60° + \alpha) + \sin(30° + \alpha) = \cos\alpha$.

(*b*) $\sin(60° + \theta) - \sin(60° - \theta) = \sin\theta$.

(*c*) $\cos(30° + \theta) - \cos(30° - \theta) = -\sin\theta$.

(*d*) $\cos(45° + \theta) + \cos(45° - \theta) = \sqrt{2}\cdot\cos\theta$.

(*e*) $\sin\left(\alpha + \frac{\pi}{3}\right) + \sin\left(\alpha - \frac{\pi}{3}\right) = \sin\alpha$.

(*f*) $\cos\left(\alpha + \frac{\pi}{6}\right) + \cos\left(\alpha - \frac{\pi}{6}\right) = \sqrt{3}\cdot\cos\alpha$.

(*g*) $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$. (*h*) $\tan(45° - \theta) = \dfrac{1 - \tan\theta}{1 + \tan\theta}$.

12. By using the functions of 60° and 30° find the value of sin 90°; cos 90°.

13. Find in radical form the value of sin 15°; cos 15°; tan 15°; sin 105°; cos 105°; tan 105°.

14. If $\tan\alpha = \frac{4}{3}$, $\sin\beta = \frac{5}{13}$, and α is in the third quadrant while β is in the second, find $\sin(\alpha \pm \beta)$; $\cos(\alpha \pm \beta)$; $\tan(\alpha \pm \beta)$.

Prove the following identities:

15. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta}$.

16. $\dfrac{\sin 2\alpha}{\sec\alpha} + \dfrac{\cos 2\alpha}{\csc\alpha} = \sin 3\alpha$.

17. $\dfrac{\tan\alpha - \tan(\alpha - \beta)}{1 + \tan\alpha\tan(\alpha - \beta)} = \tan\beta$.

18. $\tan(\theta \pm 45°) + \text{ctn}(\theta \mp 45°) = 0$.

19. (*a*) $\sin(180° - \theta) = \sin\theta$.
(*b*) $\cos(180° - \theta) = -\cos\theta$.
(*c*) $\tan(180° - \theta) = -\tan\theta$.

20. $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$.

21. $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

22. $\text{ctn}(\alpha+\beta) = \dfrac{\text{ctn}\,\alpha\cot\beta - 1}{\text{ctn}\,\alpha + \text{ctn}\,\beta}$.

23. $\text{ctn}(\alpha-\beta) = \dfrac{\text{ctn}\,\alpha\,\text{ctn}\,\beta + 1}{\text{ctn}\,\beta - \text{ctn}\,\alpha}$.

24. Prove Arc tan $\frac{1}{2}$ + Arc tan $\frac{1}{3} = \pi/4$

[HINT: Let Arc tan $\frac{1}{2} = x$ and Arc tan $\frac{1}{3} = y$. Then we wish to prove $x + y = \pi/4$, which is true since $\tan(x + y) = 1$.]

25. Prove Arc sin a + Arc cos $a = \frac{\pi}{2}$, if $0 < a < 1$.

26. Prove Arc sin $\frac{8}{17}$ + Arc sin $\frac{3}{5}$ = Arc sin $\frac{77}{85}$.

27. Prove Arc tan 2 + Arc tan $\frac{1}{2} = \pi/2$.

28. Prove Arc cos $\frac{3}{5}$ + Arc cos $(-\frac{5}{13})$ = Arc cos $(-\frac{63}{65})$.

29. Prove Arc tan $\frac{8}{15}$ + Arc tan $\frac{3}{4}$ = Arc tan $\frac{77}{36}$.

30. Find the value of sin [Arc sin $\frac{4}{5}$ + Arc ctn $\frac{4}{3}$].

31. Find the value of sin [Arc sin a + Arc sin b] if $0 < a < 1$, $0 < b < 1$.

32. Expand $\sin(x+y+z)$; $\cos(x+y+z)$.
[HINT: $x+y+z=(x+y)+z$.]

33. The area A of a triangle was computed from the formula $A=\frac{1}{2}ab\sin\theta$. If an error ϵ was made in measuring the angle θ, show that the corrected area A' is given by the relation $A'=A(\cos\epsilon+\sin\epsilon\operatorname{ctn}\theta)$.

139. Functions of Double Angles. In this and the following articles (§§ 139–141) we shall derive from the addition formulas a variety of other relations which are serviceable in transforming trigonometric expressions. Since the formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ are true for all angles α and β, they will be true when $\beta=\alpha$. Putting $\beta=\alpha$, we obtain

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{1}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha. \tag{2}$$

Since $\sin^2\alpha+\cos^2\alpha=1$, we have also

$$\cos 2\alpha = 1 - 2\sin^2\alpha \tag{3}$$

$$= 2\cos^2\alpha - 1. \tag{4}$$

Similarly the formula for $\tan(\alpha+\beta)$ (which is true for all angles α, β, and $\alpha+\beta$ which have tangents) becomes, when $\beta=\alpha$,

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}, \tag{5}$$

which holds for every angle for which both members are defined.

The above formulas should be learned in words. For example, formula (1) states that the sine of any angle equals twice the sine of half the angle times the cosine of half the angle. Thus

$$\sin 6x = 2\sin 3x\cos 3x,$$

$$\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x},$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}.$$

140. Functions of Half Angles. From (3), § 139, we have

$$2 \sin^2 \frac{\alpha}{2} = 1 - \cos \alpha.$$

Therefore

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}. \tag{6}$$

From (4), § 139, we have

$$2 \cos^2 \frac{\alpha}{2} = 1 + \cos \alpha.$$

Therefore

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}. \tag{7}$$

Formulas (6) and (7) are at once seen to hold for all angles α. Now, if we divide formula (6) by formula (7), we obtain

$$\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}, \tag{8}$$

which is true for all angles α except $n \cdot 180°$, where n is any odd integer.

Example. Given $\sin A = -3/5$, $\cos A$ negative; find $\sin (A/2)$.

Since the angle A is in the third quadrant, $A/2$ is in the second or fourth quadrant, and hence $\sin (A/2)$ may be either positive or negative. Therefore, since $\cos A = -4/5$, we have

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 + \frac{4}{5}}{2}} = \pm \frac{3}{\sqrt{10}} = \pm \frac{3}{10} \sqrt{10}.$$

EXERCISES

Complete the following formulas and state whether they are true for all angles:

1. $\sin 2\alpha =$

2. $\cos 2\alpha =$ (three forms).

3. $\tan 2\alpha =$

4. $\sin \frac{\alpha}{2} =$

5. $\cos \frac{\alpha}{2} =$

6. $\tan \frac{\alpha}{2} =$

7. In what quadrant is $\theta/2$ if θ is positive, less than 360°, and in the second quadrant? third quadrant? fourth quadrant?

8. Express $\cos 2\alpha$ in terms of $\cos 4\alpha$.

9. Express $\sin 6x$ in terms of functions of $3x$.

10. Express $\tan 4\alpha$ in terms of $\tan 2\alpha$.

11. Express $\tan 4\alpha$ in terms of $\cos 8\alpha$.

12. Express $\sin x$ in terms of functions of $x/2$.

13. Explain why the formulas for $\sin x$ and $\cos x$ in terms of functions of $2x$ have a double sign.

14 From the functions of 30° find those of 60°.

15. From the functions of 60° find those of 30°.

16. From the functions of 30° find those of 15°.

17. From the functions of 15° find those of 7.5°.

18. Find the functions of 2α if $\sin\alpha = \frac{3}{5}$ and α is in the second quadrant.

19. Find the functions of $\alpha/2$ if $\cos\alpha = -0.6$ and α is in third quadrant, positive, and less than 360°.

20. Express $\sin 3\alpha$ in terms of $\sin\alpha$. [Hint: $3\alpha = 2\alpha + \alpha$.]

21. From the value of cos 45° find the functions of 22.5°.

22. Given $\sin\alpha = \dfrac{5}{13}$ and α in the second quadrant. Find the values of

(*a*) $\sin 2\alpha$. (*c*) $\cos 2\alpha$. (*e*) $\tan 2\alpha$.

(*b*) $\sin\dfrac{\alpha}{2}$. (*d*) $\cos\dfrac{\alpha}{2}$. (*f*) $\tan\dfrac{\alpha}{2}$.

23. If $\tan 2\alpha = \dfrac{3}{4}$ find $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ if α is an angle in the third quadrant.

Prove the following identities:

24. $\dfrac{1+\cos\alpha}{\sin\alpha} = \operatorname{ctn}\dfrac{\alpha}{2}$.

25. $\left[\sin\dfrac{\theta}{2} - \cos\dfrac{\theta}{2}\right]^2 = 1 - \sin\theta$.

26. $\dfrac{\cos 2\theta + \cos\theta + 1}{\sin 2\theta + \sin\theta} = \operatorname{ctn}\theta$.

27. $\dfrac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} = \tan\theta$.

28. $\sin\dfrac{\alpha}{2} + \cos\dfrac{\alpha}{2} = \pm\sqrt{1 + \sin\alpha}$.

29. $\sec\alpha + \tan\alpha = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)$

30. $2 \operatorname{Arc}\cos x = \operatorname{Arc}\cos(2x^2 - 1)$.

31. $2 \operatorname{Arc}\cos x = \operatorname{Arc}\sin(2x\sqrt{1-x^2})$.

32. $\tan\,[2\text{ Arc}\tan x] = \dfrac{2x}{1-x^2}$.

33. $\cos\,[2\text{ Arc}\tan x] = \dfrac{1-x^2}{1+x^2}$.

34. $\tan\,[2\text{ Arc}\sec x] = \pm\dfrac{2\sqrt{x^2-1}}{2-x^2}$.

35. $\cos\,(2\text{ Arc}\sin a) = 1 - 2a^2$.

Solve the following equations:

36. $\cos 2x + 5\sin x = 3$.

37. $\cos 2x - \sin x = \frac{1}{2}$.

38. $\sin 2x \cos x = \sin x$.

39. $2\sin^2 x + \sin^2 2x = 2$.

40. $\sin^2 2x - \sin^2 x = \frac{5}{4}$.

41. $\sin 2x = 2\cos x$.

42. $2\sin^2 2x = 1 - \cos 2x$.

43 $\operatorname{ctn} x - \csc 2x = 1$.

44. A flagpole 50 ft. high stands on a tower 49 ft. high. At what distance from the foot of the tower will the flagpole and the tower subtend equal angles?

45. The dial of a town clock has a diameter of 10 ft. and its center is 100 ft. above the ground. At what distance from the foot of the tower will the dial be most plainly visible? [The angle subtended by the dial must be as large as possible.]

141. Product Formulas.

From § 138 we have

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Adding, we get

$$(1)\qquad \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta.$$

Subtracting, we have

$$(2)\qquad \sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta.$$

Now, if we let $\alpha+\beta = P$ and $\alpha-\beta = Q$,

then $$\alpha = \frac{P+Q}{2},\quad \beta = \frac{P-Q}{2}.$$

Therefore formulas (1) and (2) become

$$\mathbf{\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2},}$$

$$\mathbf{\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}.}$$

Similarly, starting with $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ and performing the same operations, the following formulas result:

$$\cos P + \cos Q = 2 \cos \frac{P+Q}{2} \cos \frac{P-Q}{2},$$

$$\cos P - \cos Q = -2 \sin \frac{P+Q}{2} \sin \frac{P-Q}{2}.$$

In words:

the sum of two sines =
twice sin (half sum) times cos (half difference),

the difference of two sines =
twice cos (half sum) times sin (half difference),*

the sum of two cosines =
twice cos (half sum) times cos (half difference),

the difference of two cosines =
minus twice sin (half sum) times sin (half difference).*

Example 1. Prove that

$$\frac{\cos 3x + \cos x}{\sin 3x + \sin x} = \text{ctn } 2x,$$

for all angles for which both members are defined.

$$\frac{\cos 3x + \cos x}{\sin 3x + \sin x} = \frac{2 \cos \frac{1}{2}(3x + x) \cos \frac{1}{2}(3x - x)}{2 \sin \frac{1}{2}(3x + x) \cos \frac{1}{2}(3x - x)} = \frac{\cos 2x}{\sin 2x} = \text{ctn } 2x.$$

Example 2. Reduce $\sin 4x + \cos 2x$ to the form of a product.

We may write this as $\sin 4x + \sin(90° - 2x)$, which is equal to

$$2 \sin \frac{4x + 90° - 2x}{2} \cos \frac{4x - 90° + 2x}{2} = 2 \sin(45° + x) \cos(3x - 45°).$$

EXERCISES

Reduce to a product:

1. $\sin 4\theta - \sin 2\theta$.
2. $\cos \theta + \cos 3\theta$.
3. $\cos 6\theta + \cos 2\theta$.
4. $\cos 2\theta + \sin 2\theta$.
5. $\cos 3\theta - \cos 6\theta$.
6. $\sin(x + \Delta x) - \sin x$.
7. $\cos 3x + \sin 5x$.
8. $\sin 20° - \sin 60°$.

Show that

9. $\sin 20° + \sin 40° = \cos 10°$.
10. $\cos 50° + \cos 70° = \cos 10°$.
11. $\dfrac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \tan 30°$.
12. $\dfrac{\sin 15° + \sin 75°}{\sin 15° - \sin 75°} = -\tan 60°$.
13. $\dfrac{\sin 3\theta - \sin 5\theta}{\cos 3\theta - \cos 5\theta} = -\text{ctn } 4\theta$.

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* The difference is taken, first angle minus the second.

Prove the following identities:

14. $\dfrac{\sin 4\alpha + \sin 3\alpha}{\cos 3\alpha - \cos 4\alpha} = \operatorname{ctn}\dfrac{\alpha}{2}.$

15. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \dfrac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}.$

16. $\dfrac{\cos\alpha + 2\cos 3\alpha + \cos 5\alpha}{\cos 3\alpha + 2\cos 5\alpha + \cos 7\alpha} = \dfrac{\cos 3\alpha}{\cos 5\alpha}.$

17. $\dfrac{\cos\alpha - \cos\beta}{\cos\alpha + \cos\beta} = -\dfrac{\tan\frac{1}{2}(\alpha+\beta)}{\operatorname{ctn}\frac{1}{2}(\alpha-\beta)}.$

18. $\dfrac{\sin(n-2)\theta + \sin n\theta}{\cos(n-2)\theta - \cos n\theta} = \operatorname{ctn}\theta.$

Solve the following equations:

19. $\cos\theta + \cos 5\theta = \cos 3\theta.$

20. $\sin\theta + \sin 5\theta = \sin 3\theta.$

21. $\sin 3\theta + \sin 7\theta = \sin 5\theta.$

22. $\sin 4\theta - \sin 2\theta = \cos 3\theta.$

23. $\cos 7\theta - \cos\theta = -\sin 4\theta.$

142. Law of Tangents. A method for shortening computation will be presented in the next chapter. In applying this method to the solution of triangles the formulas given below are valuable. We shall state first the so-called ***law of tangents:***

The difference of two sides of a triangle is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

PROOF.
$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Hence, by proportion, we have
$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

But
$$\frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}}{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}} = \frac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}}.$$

Therefore
$$\frac{a-b}{a+b} = \frac{\tan\dfrac{A-B}{2}}{\tan\dfrac{A+B}{2}}.$$

143. Angles of a Triangle in Terms of the Sides. Construct the inscribed circle of the triangle and denote its radius by r. If the perimeter $a+b+c=2s$, then (Fig. 130)

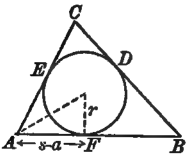

FIG. 130.

$$AE = AF = s - a.$$
$$BD = BF = s - b.$$
$$CD = CE = s - c.$$

Then $\tan\frac{1}{2}A = \frac{r}{s-a}$, $\tan\frac{1}{2}B = \frac{r}{s-b}$, $\tan\frac{1}{2}C = \frac{r}{s-c}$,
where, from § 130,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

MISCELLANEOUS EXERCISES

1. Reduce to radians 65°, $-135°$, $-300°$, 20°.

2. Reduce to degrees π, 3π, -2π, 4π radians.

3. Find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$ when it is given that α and β are positive and acute and $\tan\alpha = \frac{3}{4}$ and $\sec\beta = \frac{13}{5}$.

4. Find $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$ when it is given that $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$.

5. Prove that $\sin 4\alpha = 4\sin\alpha\cos\alpha - 8\sin^3\alpha\cos\alpha$.

6. Given $\sin\theta = \frac{2}{\sqrt{5}}$, and θ in the second quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

Prove the following identities:

7. $\sin 2\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha}$.

8. $\cos 2\alpha = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

9. $\sec 2\alpha = \dfrac{\csc^2\alpha}{\csc^2\alpha - 2}$.

10. $\tan\alpha = \dfrac{\sin 2\alpha}{1+\cos 2\alpha}$.

11. $\sin(\alpha+\beta)\cos\beta - \cos(\alpha+\beta)\sin\beta = \sin\alpha$.

12. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$, if $\alpha + \beta + \gamma = 180°$.

13. $\dfrac{\cos\alpha}{1-\sin\alpha} = \dfrac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$.

14. $1 + \tan\alpha \tan\dfrac{\alpha}{2} = \sec\alpha.$

15. $\dfrac{\sin^3\alpha + \cos^3\alpha}{\sin\alpha + \cos\alpha} = \dfrac{2 - \sin 2\alpha}{2}.$

16. $\dfrac{\sin 4\alpha}{\sin 2\alpha} = 2\cos 2\alpha.$

17. $\text{Arc}\cos\frac{4}{5} + \text{Arc}\tan\frac{3}{5} = \text{Arc}\tan\frac{27}{11}.$

Solve the following equations:

18. $\cos 2\alpha = \cos^2 a.$

19. $2\sin\alpha = \sin 2\alpha.$

20. $\cos 2\alpha + \cos\alpha = -1.$

21. $\sin\alpha + \sin 2\alpha + \sin 3\alpha = 0.$

22. $\sin 2\alpha - \cos 2\alpha - \sin\alpha + \cos\alpha = 0.$

23. $\text{Arc}\tan x + \text{Arc}\tan(1 - x) = \text{Arc}\tan\frac{4}{3}.$

24. $\text{Arc}\sin x + \text{Arc}\sin 2x = \dfrac{\pi}{3}.$

25. $\text{Arc}\tan\dfrac{x+1}{x-1} + \text{Arc}\tan\dfrac{x-1}{x} = 180° + \text{Arc}\tan(-7).$

26. $\text{Arc}\sin x + \text{Arc}\sin\dfrac{x}{2} = 120°.$

27. $\text{Arc}\sin x + 2\,\text{Arc}\cos x = \dfrac{2\pi}{3}.$

In a right triangle ABC, right angled at C, prove

28. $\sin^2\dfrac{B}{2} = \dfrac{c-a}{2c}.$ **29.** $\cos^2\dfrac{A}{2} = \dfrac{b+c}{2c}.$ **30.** $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}.$

31. Solve for x and y the following equations:

$$x\sin\alpha + y\cos\alpha = \sin\alpha,$$
$$x\cos\alpha - y\sin\alpha = \cos\alpha.$$

32. Solve for x and y the following equations:

$$x\cos\theta - y\sin\theta = \sin\theta,$$
$$x\sin\theta + y\cos\theta = \cos\theta.$$

33. If $2x$ is less than 90° and $\sin x = \cos(2x + 40°)$, find the value of x.

34. Find α so that the equation $x^2 + 2x\cos\alpha + 1 = 0$ shall have equal roots.

35. Find α so that the equation $3x^2 + 2x\sec\alpha + 1 = 0$ shall have equal roots.

CHAPTER VIII

THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS

144. The Invention of Logarithms. In the last two chapters we have had occasion to do a considerable amount of numerical computation. In spite of the fact that we have confined these computations to comparatively small numbers and have had the assistance of tables of squares and square roots, the calculations have often been laborious.

To carry out by the methods thus far at our disposal the computations involved in many of the problems of insurance, engineering, astronomy, etc., would require a prohibitive amount of labor. That it is now practicable to effect such computations is largely due to the invention of ***logarithms*** by John Napier (1550–1617), Baron of Merchiston, in Scotland.

As in the case of many epoch-making inventions, the fundamental idea of Napier was extraordinarily simple. It may be explained as follows. Consider the function $y = 2^x$. We readily obtain the following table of corresponding values:

(1)

x	1	2	3	4	5	6	7	8	9	10	11	12
$y = 2^x$	2	4	8	16	32	64	128	256	512	1024	2048	4096

Now, since $2^u \cdot 2^v = 2^{u+v}$, it is clear that, if we desire to obtain the product of two numbers in the lower line of the table, we need only add the two corresponding numbers in the upper line (the exponents), and then find the number in the lower line which corresponds to this sum. For example, to find the product of 128 × 16, we find from the table that the numbers corresponding to 128 and 16 are 7 and 4, respectively; the sum of the last pair is 11 and the number in the lower line corresponding to 11 is 2048, which is the product sought. Or again, to find 4096 ÷ 512, we find the corresponding exponents 12 and 9 in the table, subtract ($12 - 9 = 3$), and find the required quotient to be 8. How would you justify the latter procedure?

While the fundamental idea here described is simple, considerable insight was required to make the idea practicable. For, the above table makes possible the finding of the product of two numbers only when the numbers in question and their product are to be found in the lower line of the table. In order to be useful in practical computation it is obviously necessary to construct a table which will contain every number, or at least from which the corresponding "exponent" of any number can easily be obtained either precisely or with a high degree of approximation. The problem confronting Napier was to *fill in the gaps* in the numbers of the lower line of the table on p. 212, while preserving the ***fundamental property*** of the table, viz. that *to the product of any two numbers of the lower line corresponds the sum of the two corresponding numbers of the upper line.*

145. Extension of the Table.

An examination of table (1) reveals the following properties: (*a*) the values of x form an arithmetic progression (A.P.), since every number after the first is obtained by adding 1 to the preceding number; (*b*) the values of y form a geometric progression (G.P.), since every number after the first is obtained by multiplying the preceding number by 2. These considerations suggest the possibility of extending the table in two ways.

In the first place, we may extend it to the left so as to make the lower line contain numbers less than 2. To do this, we need only subtract 1 successively from the numbers of the upper line and divide by 2 successively the numbers of the lower line. We then obtain a table extending in both directions:

(2)

-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0.03125	0.0625	0.125	0.25	0.5	1	2	4	8	16	32	64	128

This table is still satisfactory. If we desire to multiply 128 by 0.0625, we add the corresponding numbers of the upper line, namely, 7 and -4; thus we obtain the number 3, which according to the previous rule should give $128 \times 0.0625 = 8$, which is correct. That the rule still applies may be tested on other products; the fact that it *does* will be proved later.

In the second place we may find new numbers to fill the gaps in the original table, by inserting arithmetic means between the successive values of x and geometric means between the successive values of y. Thus, if we take the following portion of the preceding table

-2	-1	0	1	2	3	4
$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

and insert between every two successive numbers of the upper line their arithmetic, and between every two successive numbers of the lower line their geometric mean, we obtain the table

(3)

-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
$\frac{1}{4}$	$\frac{1}{4}\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	1	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$	8	$8\sqrt{2}$	16

If the radicals are expressed approximately as decimals, this table takes the form

– 2.0	– 1.5	– 1.0	– 0.5	0	0.5	1.0	1.5	2	2.5	3	3.5	4
0.25	0.35	0.50	0.72	1.00	1.41	2.00	2.83	4.00	5.66	8.00	11.31	16

Repeating this process of inserting means, we get the following table. To save space, we have begun the arithmetic progression with 0 and the geometric progression with 1, and have not carried the table as far as in the preceding case.

(4)

x (*A.P.*)	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
y (*G.P.*)	1.00	1.19	1.41	1.68	2.00	2.38	2.83	3.36	4.00	4.76

The rule for multiplying two values of y seems to apply also to this table, at least approximately. For example, if we apply the rule to find 3.36×1.19, we note that the sum of the corresponding values of x is $1.75 + 0.25 = 2.00$ and conclude that $3.36 \times 1.19 = 4.00$. Actual multiplication gives $3.36 \times 1.19 = 3.9984$. The discrepancy we may attribute to the fact that the values of y other than 1, 2, 4 are only approximations to the true values.*

The process used in constructing this table may be continued indefinitely. It enables us to *interpolate* a new value of x between any two successive values of x and a new value of y between the two corresponding values of y. But this means that we can make the values of x and y as *dense* as we please, in other words, we can make the difference between successive values of y as small as we please. By continuing the process long enough we can make any number appear among the values of y to as high a degree of approximation as we desire and our intention of *filling the gaps* will then be attained. We must now prove, however, that the rule for multiplication does really hold in the extended table. Thus far we have merely verified this rule for special cases.

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* In fact the rules for computing with approximate numbers would lead us to write 4.00 in place of 3.9984 as we have no right to retain more than two decimal places. See § 160.

EXERCISES

1. Assuming that the rule for multiplication applies, find by means of table (4) the following products.

$$3.36 \times 1.41,\ 1.68 \times 2.38,\ (1.68)^2,\ (1.19)^5.$$

Check by ordinary multiplication.

146. Arithmetic and Geometric Progressions. The tables constructed consist of an arithmetic progression one term of which is the number 0 (the terms of this arithmetic progression we denoted by x) and a geometric progression one term of which is the number 1 (the terms of this geometric progression we denoted by y). Moreover, to every value of x corresponds a definite value of y in such a way that to $x = 0$ corresponds $y = 1$, and that to each succeeding (or preceding) value of x corresponds the succeeding (or preceding) value of y. Now suppose that the common difference of the arithmetic progression is d and that the common ratio of the geometric progression is r. The correspondence between the values of x and y would then be exhibited in the following table.

(5)

x	$\cdots$	$-md$	$\cdots$	$-3d$	$-2d$	$-d$	0	d	$2d$	$3d$	$\cdots$	nd	$\cdots$
y	$\cdots$	$\frac{1}{r^m}$	$\cdots$	$\frac{1}{r^3}$	$\frac{1}{r^2}$	$\frac{1}{r}$	1	r	r^2	r^3	$\cdots$	r^n	$\cdots$

We shall now prove that *in this table, to the product of any two values of y corresponds the sum of the two corresponding values of x.*

If the two values of y are both to the right of $y = 1$, for example $y_1 = r^p$, $y_2 = r^q$, then the corresponding values of x are pd and qd. To the product $y_1y_2 = r^pr^q$ corresponds $(p+q)\,d$. If the two values of y are both to the left of $y = 1$, the proof is similar. It is left as an exercise.

If one value is to the left of $y = 1$, for example, $y = 1/r^p$, and the other value is to the right, for example $y_2 = r^q$, the corresponding values of x are $-pd$ and qd respectively. The product y_1y_2 is equal to $(1/r^p)\,r^q = r^{q-p}$ if $q > p$, and is equal to $1/r^{p-q}$ if $q < p$. The value of x corresponding to y_1y_2 is then $(q-p)\,d$, if $q > p$ and $-(p-q)d$ if $q < p$. But $(q-p)d = -(p-q)d = qd + (-pd)$. The discussion of the case $p = q$ is left as an exercise. If one of the values of y is 1, the desired result follows immediately. Why?

In view of this theorem the validity of the rule used in the last article for multiplication is established. For tables (2) and (4) are both tables of the type (5), the former having $d = 1$ and $r = 2$, the latter having $d = 0.25$ and $r = \sqrt[4]{2} = 1.19$ (approximately).

147. The Exponential Function $a^x (a > 0)$. Let us now consider the table

x	$\cdots$	$-m$	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$	n	$\cdots$
y	$\cdots$	$\frac{1}{a^m}$	$\cdots$	$\frac{1}{a^3}$	$\frac{1}{a^2}$	$\frac{1}{a}$	1	a	a^2	a^3	$\cdots$	a^n	$\cdots$

where a represents any *positive* number.* This table defines y as a function of x. Morover, this table is a table of the type (5); and all tables obtained by interpolating arithmetic means between two successive values of x and the same number of geometric means between the corresponding values of y are of the type (5). Thus if we interpolate q arithmetic means between $x = 0$ and $x = 1$, and q geometric means between $y = 1$ and $y = a$, we obtain the following table:

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* The value $a = 1$ leads to trivial results. Hence, we assume also that $a \neq 1$.

x	...	$-\frac{p}{q}$	...	-1	...	$\frac{-2}{q}$	$-\frac{1}{q}$	0	$\frac{1}{q}$	$\frac{2}{q}$	...	$\frac{q-1}{q}$	1	$\frac{q+1}{q}$	...	2	...	$\frac{p}{q}$	...
y	...	$\frac{1}{(\sqrt[q]{a})^p}$	...	$\frac{1}{a}$	...	$\frac{1}{(\sqrt[q]{a})^2}$	$\frac{1}{\sqrt[q]{a}}$	1	$\sqrt[q]{a}$	$(\sqrt[q]{a})^2$	...	$(\sqrt[q]{a})^{q-1}$	a	$(\sqrt[q]{a})^{q+1}$	...	a^2	...	$(\sqrt[q]{a})^p$	...

which is a table of the type (5), with $d = 1/q$ and $r = \sqrt[q]{a}$.*

The function y of x thus defined is $y = a^x$, for $x = 1, 2, 3, \cdots$. We are therefore led to *define* the expression a^x for fractional and negative values of x and for $x = 0$ as follows:

(1) $a^0 = 1$.

(2) **$a^{1/q}$ means $\sqrt[q]{a}$, where q is a positive integer.**

(3) **$a^{p/q}$ means $(\sqrt[q]{a})^p$, or its equal $\sqrt[q]{a^p}$, where p and q are positive integers.**

(4) **a^{-n} means $1/a^n$, where n is any positive rational number.**

In view of the fundamental property of any table of type (5), whereby to the product of any two values of y corresponds the sum of the two corresponding values of x, we have

$$a^u \cdot a^v = a^{u+v}$$

for all values of u and v for which the expressions a^u, a^v, and a^{u+v} have been defined.

The function $y = a^x\,(a > 0)$ has now been defined for all *rational* values of x. To complete the definition of this function for all real values of x, we must indicate the meaning of a^x when x is an *irrational number.* To carry this definition through in all its details is beyond the scope of an elementary course. But we have seen that any irrational number may be represented approximately by a rational number, with an error as small as we please. (See § 29.) Thus $\sqrt{3}$ is represented approximately by the rational numbers 1.7, 1.73, 1.732, …. Our previous definitions have given a definite meaning, for example, to $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, …. The values of the latter expressions are *by definition* approximate values of $2^{\sqrt{3}}$. We take for granted without proof the fact that the successive numbers

---

* We should keep in mind that the symbol $\sqrt[q]{a}$ $(a>0)$ means the positive qth root of a. Thus $\sqrt[4]{16} = 2$, not -2.

$$(6) \qquad 2^{1.7},\ 2^{1.73},\ 2^{1.732},\ \dots,$$

as the exponents represent closer and closer approximations to $\sqrt{3}$, approach closer and closer to a definite number. This definite number is by definition the value of $2^{\sqrt{3}}$. Similar considerations apply to the definition of a^x, where a is any positive number and x is any irrational number. The principle involved is briefly expressed as follows:

An approximate value of x gives an approximate value of a^x. The value of a^x can be found as accurately as we please by using a sufficiently accurate approximation to x.

The objection might be raised that the calculation of $2^{1.7}$ involves the extraction of the 10th root of 2 and the calculation of $2^{1.78}$ involves the extraction of the 100th root of 2, etc., and perhaps we do not know how to extract these roots. As a matter of fact we can calculate $2^{\sqrt{3}}$ as accurately as we please by extracting square roots only. The process is as follows: We know that $\sqrt{3} = 1.7320$ accurately to four decimal places. Now by table (4), p. 215, we see that $2^{1.50} = 2.83$ and $2^{1.75} = 3.36$. We carry the computation to more places and have $2^{1.5000} = 2.8284$ and $2^{1.7500} = 3.3635$.

Now, 1.7320 lies between 1.5000 and 1.7500, the arithmetic mean of which is 1.6250. The geometric mean of 2.8384 and 3 3635 is 3.0844. According to our previous definitions we have then $2^{1.6250} = 3.0844$.

Inserting means between the last two results we have $2^{1.6875} = 3.2209$. By inserting arithmetic means between the properly selected exponents and geometric means between the corresponding powers of 2 we can ultimately obtain the value of $2^{1.7320}$. The results of the necessary steps are:
$2^{1.7188} = 3.2915$, $2^{1.7344} = 3.3274$, $2^{1.7266} = 3.3094$, $2^{1.7305} = 3.3182$,
$2^{1.7325} = 3.3228$, $2^{1.7315} = 3.3205$, $2^{1.7320} = 3.3217$.
The process here illustrated makes it possible to calculate $2^{\sqrt{3}}$ to as high a degree of approximation as we please, since we can carry the computation to as large a number of decimal places as we please.

148. The Laws of Exponents. The function $y = a^x$ $(a > 0)$ is now defined for all *real* values of x. This function is called the ***exponential function of base a.*** The laws of exponents

$$\left.\begin{array}{lc} \text{I.} & a^u \cdot a^v = a^{u+v} \\ \text{II.} & (a^u)^v = a^{uv} \\ \text{III.} & a^u \cdot b^u = (ab)^u \end{array}\right\}, \qquad a > 0,\ b > 0,$$

which were derived previously (§ 42) for positive integral exponents, hold for all real values of u and v. The first of these we have already derived. The last two may be readily proved for negative, fractional, and zero exponents by using the definition of a^x.

Thus by definition, if $u = p/q$ and $v = n$, where p, q, n are positive integers, we have

$$(a^u)^v = (a^{\frac{p}{q}})^n = ((\sqrt[q]{a})^p)^n = (\sqrt[q]{a})^{pn} = a^{\frac{pn}{q}} = a^{uv}.$$

If u is any positive rational number and $v = p/q$, where p, q are positive integers, we have,

$$(a^u)^v = (a^u)^{\frac{p}{q}} = \sqrt[q]{(a^u)^p} = \sqrt[q]{a^{up}} = a^{\frac{up}{q}} = a^{uv}.$$

If $u = -n$, where n is a positive rational number, and if $v = p/q$, where p and q are positive integers, we have

$$(a^u)^v = (a^{-n})^{p/q} = \left(\sqrt[q]{\frac{1}{a^n}}\right)^p = \frac{1}{(\sqrt[q]{a^n})^p} = \frac{1}{a^{\frac{np}{q}}} = a^{-\frac{np}{q}} = a^{uv}.$$

If u is any rational number and $v = -n$, where n is any positive rational number,

$$(a^u)^v = (a^u)^{-n} = \frac{1}{(a^u)^n} = \frac{1}{a^{un}} = a^{-un} = a^{uv}.$$

If either u or v is zero, the result is immediate. Hence the law II is proved for rational exponents.

A similar proof of the law III is left as an exercise.

149. The Graph of the Exponential Function. Figure 131 represents the graph of the function $y = 2^x$, drawn from the tabular representation given in the first table on p. 215.

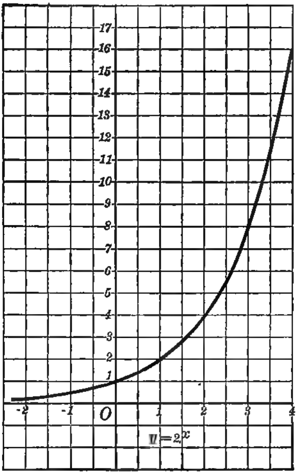

FIG. 131

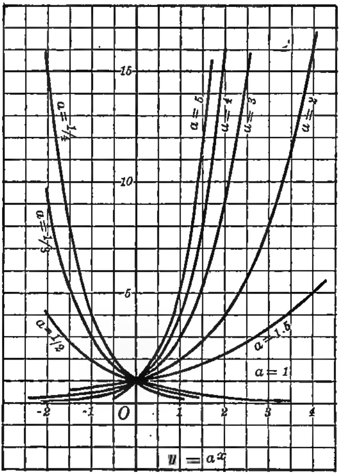

FIG. 132

It will be noted that all curves of the system $y = a^x$ pass through the point (0, 1). By hypothesis $a > 0$. If $a > 1$, the function a^x is an increasing function; while if $a < 1$, the function is a decreasing function. Figure 132 shows some of the curves of the system $y = a^x$.

EXERCISES

1. Calculate the value of the exponential function 3^x
(*a*) for the values of $x = 1, 2, 3, 4, 0, -1, -2, -3, -4$;
(*b*) for the values $x = 0.5, 1.5, 2.5, 3.5, -0.5, -1.5, -2.5$;
(*c*) for the values $x = 0.25, 0.75, 1.25, 1.75$.
Arrange the results in the form of a single table.

2. Show how to use the table constructed in Ex. 1 to solve problems in multiplication, division, raising to powers, and extracting roots. Make up your own problems and check your results by the methods of arithmetic.

3. Describe in detail how you would find the value of $3^{\sqrt{2}}$. Between what two numbers in the table found in Ex. 1 does the value of $3^{\sqrt{2}}$ lie?

4. Construct the graph of $y = 3^x$ for values of x between -2 and 3.

5. What is meant by $a^{\frac{3}{2}}$? $x^{\frac{2}{3}}$? $(1/y)^{\frac{1}{3}}$?

6. What is the value of $8^{\frac{1}{3}}$? $27^{\frac{2}{3}}$? $(0.001)^{\frac{1}{3}}$? $(\frac{1}{2})^3$?

7. Simplify $(18)^{\frac{1}{2}} \div (3)^{\frac{1}{2}}$.

8. Perform the following indicated operations:

(*a*) $(x^{\frac{3}{4}})^{\frac{5}{3}}$.

(*b*) $(a^{\frac{3}{2}}b^{\frac{2}{3}}c^{\frac{1}{4}})^{\frac{1}{2}}$.

(*c*) $(32\,x^0y^{10})^{\frac{1}{5}}$.

(*d*) $\left(\dfrac{a^9b^{-3}}{x^6y^{-\frac{3}{2}}}\right)^{-\frac{1}{3}}$.

(*e*) $(a^{-1} + b^{-1})^2$.

(*f*) $(2^{\frac{3}{2}})^4$.

9. Multiply

(*a*) $(a^{-1} + a)(a^{-1} - a)$.

(*b*) $(a^{-1} - a^0)(a^{-2} - a^0)(a^{-3} - a^0)$.

(*c*) $(x^{\frac{1}{6}} - y^{\frac{1}{6}})(x^{\frac{5}{6}} - y^{\frac{5}{6}})$.

(*d*) $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.

(*e*) $(a^{\frac{3}{x}} - 2\,a^{\frac{2}{x}} + 3\,a^{\frac{1}{x}})(2\,a^{\frac{2}{x}} - a^{\frac{1}{x}} + 2)$.

(*f*) $(y^{\frac{3}{n}} - ay^{\frac{2}{n}} + 3\,by^{\frac{1}{n}} - c)(y^{\frac{2}{n}} + by^{\frac{1}{n}} - cy^0)$.

10. Divide

(*a*) $(x+1)$ by $(\sqrt[6]{x} + 1)$.

(*b*) $(x^{\frac{1}{6}} - y^{\frac{1}{6}})$ by $(x^{\frac{1}{18}} - y^{\frac{1}{18}})$.

(*c*) $(a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}})$ by $(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

(*d*) $(a^{-1} + 4\,a^{\frac{1}{4}} + 6\,a^{\frac{3}{2}} + 4\,a^{\frac{11}{4}} + a^4)$ by $(a^{-\frac{1}{4}} + a)$.

11. Simplify

(*a*) $12^0 + \frac{9}{2} - 9^{-1} + \frac{1}{\sqrt[3]{64}} + 27^{\frac{2}{3}}$. (*b*) $\left(\frac{m^2p}{64\,m^{-3}p^{\frac{1}{3}}}\right)^{-\frac{1}{3}}$.

12. Simplify

$$\left[\frac{e^x + e^{-x}}{2} + \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1}\right]\left[\frac{e^x + e^{-x}}{2} - \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2 - 1}\right].$$

13. Which of the two numbers $\sqrt{5}$ and $\sqrt[3]{8}$ is the greater and why ?

14. Simplify

$$(2^{\frac{1}{2}} \times 2^{\frac{2}{3}}) \div (54)^{\frac{1}{3}}.$$

15. Prove that, if

$$p = \frac{1}{2}\left[\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} - \frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}\right],$$

then

$$(1 + p^2)^{\frac{1}{2}} = \frac{x + y}{2\sqrt{xy}}.$$

16. Reduce to simplest form

(*a*) $\sqrt[n]{\frac{4}{2^{n+2}}}$. (*b*) $\frac{ax(a^{-1}x - ax^{-1})}{x^{\frac{2}{3}} - a^{\frac{2}{3}}}$.

(*c*) $(a^4 + x^4)(a^2 - x^2)^{-\frac{3}{2}} - (a^2 - x^2)^{\frac{1}{2}}$.

150. Definition of the Logarithm. The logarithm of a number N to a base b ($b > 0$, $\neq 1$) is the exponent x of the power to which the base b must be raised to produce the number N.

That is, if

$$b^x = N,$$

then

$$x = \log_b N.$$

These two equations are of the highest importance in all work concerning logarithms. One should keep in mind the fact that if either of them is given, the other may always be inferred.

The graph of the logarithm function (Fig. 133) is obtained from the graph of the corresponding exponential function by simply turning the latter graph over about the line through the origin bisecting the first and third quadrants.

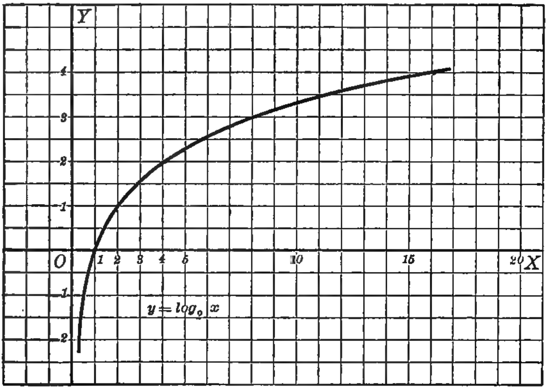

FIG. 133

EXERCISES

1. When 3 is the base what are the logarithms of 9, 27, 3, 1, 81, $\frac{1}{3}$, $\frac{1}{243}$, $27^{\frac{1}{2}}$?

2. Why cannot 1 be used as the base of a system of logarithms?

3. When 10 is the base what are the logarithms of 1, 10, 100, 1000?

4. Find the values of x which will satisfy each of the following equalities:

(*a*) $\log_3 27 = x$.
(*b*) $\log_x 3 = 1$.
(*c*) $\log_x 5 = \frac{1}{2}$.
(*d*) $\log_a a = x$.
(*e*) $\log_a 1 = x$.
(*f*) $\log_3 \frac{1}{81} = x$.
(*g*) $\log_2 x = 6$.
(*h*) $\log_{32} x = \frac{1}{5}$.
(*i*) $\log_{.001} x = .00001$.

5. Find the value of each of the following expressions:

(*a*) $\log_2 16$.
(*b*) $\log_{343} 49$.
(*c*) $\log_6 \frac{1}{216}$.
(*d*) $\log_2 \sqrt{16}$.
(*e*) $\log_{25} 125$.
(*f*) $\log_2 \frac{1}{64}$.

151. The Three Fundamental Laws of Logarithms. From the properties of the exponential function (p. 220) we derive the following fundamental laws.

I. *The logarithm of a product equals the sum of the logarithms of its factors.* Symbolically,

$$\log_b MN = \log_b M + \log_b N.$$

Proof. Let $\log_b M = x$, then $b^x = M$. Let $\log_b N = y$, then $b^y = N$. Hence we have $MN = b^{x+y}$, or

$$\log_b MN = x + y, \quad i.e. \quad \log_b MN = \log_b M + \log_b N.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.* Symbolically

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

Proof. Let $\log_b M = x$, then $b^x = M$. Let $\log_b N = y$, then $b^y = N$. Hence we have $M/N = b^{x-y}$, or

$$\log_b \frac{M}{N} = x - y, \quad i.e. \quad \log_b \frac{M}{N} = \log_b M - \log_b N$$

III. *The logarithm of the pth power of a number equals p times the logarithm of the number.* Symbolically

$$\log_b M^p = p \log_b M.$$

Proof. Let $\log_b M = x$, then $b^x = M$. Raising both sides to the pth power, we have $b^{px} = M^p$. Therefore

$$\log_b M^p = px = p \log_b M.$$

From law III it follows that the *logarithm of the real positive nth root of a number is one nth of the logarithm of the number.*

EXERCISES

1. Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, find the value of each of the following expressions:

(*a*) $\log_{10} 6$.

[HINT: $\log_{10} 2 \times 3 = \log_{10} 2 + \log_{10} 3$.]

(*b*) $\log_{10} 21.0$.

(*c*) $\log_{10} 20.0$.

(*d*) $\log_{10} 0.03$.

(*e*) $\log_{10} \frac{7}{2}$.

(*f*) $\log_{10} 5$.

[HINT: $\log_{10} 5 = \log_{10} \frac{10}{2}$]

(*g*) $\log_{10} 150$.

(*h*) $\log_{10} \sqrt{14}$.

(*i*) $\log_{10} 49$.

(*j*) $\log_{10} \sqrt{2^4 . 7^5}$.

2. Given the same three logarithms as in Ex. 1, find the value of each of the following expressions:

(*a*) $\log_{10} \frac{4 \times 5 \times 7}{32 \times 8}$.

(*b*) $\log_{10} \frac{5 \times 3 \times 20}{6 \times 7}$.

(*c*) $\log_{10} \frac{2058}{\sqrt{14}}$.

(*d*) $\log_{10} (2)^{25}$.

(*e*) $\log_{10} (3)^8 (5)^6$.

(*f*) $\log_{10} (2^8)(\frac{1}{9})$.

152. The Systems most Frequently Used. From the definition of a logarithm (§ 150) any positive number except 1 can be used as the base of a system of logarithms. As a matter of fact, however, the numbers generally used are (1) a certain irrational number which is approximately equal to 2.71828 and is denoted by e and (2) the number 10. Logarithms to the base e are important in certain theoretical problems; logarithms to this base are called ***natural.*** For numerical computation it will be seen presently that the base 10 has numerous advantages. Since different systems of logarithms are in use, it is important to know how to change from one system to another. The following law explains how this can be done.

IV. *The logarithm of a number M to the base b is equal to the logarithm of M to any base a, divided by the logarithm of b to the base a.* Symbolically,

$$\log_b M = \frac{\log_a M}{\log_a b}.$$

PROOF. Let $\log_b M = x$, then $b^x = M$. Taking the logarithms of both sides to the base a, we have

$$\log_a b^x = \log_a M, \quad \text{or} \quad x \log_a b = \log_a M,$$

i.e.

$$x = \log_b M = \frac{\log_a M}{\log_a b}.$$

153. Logarithms to the Base 10. Logarithms to the base 10 are known as ***common logarithms***, or as **Briggian logarithms,** after Henry Briggs (1556–1631) who called attention to the advantages of 10 as a base. These advantages appear below.

If 10 is the base, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc. It follows that if a number be multiplied by 10, or by any positive integral power of 10, the logarithm of the number is increased by an *integer*. In other words, the shifting to the right of the decimal point in a number changes only the integral part of the logarithm and leaves unchanged the decimal part of the logarithm.

An example will make this clear. Given $\log_{10} 2 = 0.30103$, we have $\log_{10} 20 = 1.30103$, $\log_{10} 200 = 2.30103$, etc. Or, again, given $\log_{10} 4.5607 = 0.65903$, we have $\log_{10} 45.607 = 1.65903$, $\log_{10} 456.07 = 2.65903$, $\log_{10} 4560.7 = 3.65903$, $\log_{10} 45607.0 = 4.65903$.

It should be clear from these examples that the decimal part of the logarithm of a number greater than 1 in this system depends only on the succession of figures composing the number, irrespective of where the decimal point is located; while the integral part of the logarithm of the number depends simply on the position of the decimal point.

The decimal part of a logarithm is called its ***mantissa***, the integral part its ***characteristic***. In view of what has been said above only the mantissas of logarithms to the base 10 need be tabulated. The characteristic can be found by inspection. This follows from the following considerations.

The common logarithm of a number between 1 and 10 lies between 0 and 1.

The common logarithm of a number between 10 and 100 lies between 1 and 2.

The common logarithm of a number between 100 and 1000 lies between 2 and 3.

. . . .

The common logarithm of a number between 10^n and 10^{n+1} lies between n and $n+1$.

It follows that a number with *one* digit ($\neq 0$) at the left of the decimal point has for its logarithm a number equal to $0+$ a decimal; a number with *two* digits at the left of its decimal point has for its logarithm a number equal to $1+$ a decimal; a number with *three* digits at the left of the decimal point has for its logarithm a number equal to $2+$ a decimal, etc. We conclude, therefore, that *the characteristic of the common logarithm of a number greater than* 1 *is one less than the number of digits at the left of the decimal point.*

Thus, as before, $\log_{10} 456.07 = 2.65903$.

The case of a logarithm of a number less than 1 requires special consideration. Taking the numerical example first considered above, if $\log_{10} 2 = 0.30103$, we have $\log_{10} 0.2 = 0.30103 - 1$. Why? This is a negative number, as it should be (since the logarithms of numbers less than 1 are all negative, if the base is greater than 1). But, if we were to carry out this subtraction and write $\log_{10} 0.2 = -.69897$ (which would be *correct*), it would change the mantissa, which is *inconvenient.* Hence it is customary to write such a logarithm in the form $9.30103 - 10$.

If there are n ciphers immediately following the decimal point in a number less than 1, the characteristic is $-n-1$. *For convenience, if* $n < 10$, *we write this as* $(9-n) - 10$. *This* *characteristic is written in two parts. The first part* $9 - n$ *is written at the left of the mantissa and the* -10 *at the right.*

In the sequel, unless the contrary is specifically stated we shall assume that all logarithms are to the base 10. We may accordingly omit writing the base in the symbol log when there is no danger of confusion. Thus, the equation $\log 2 = 0.30103$ means $\log_{10} 2 = 0.30103$.

154. Use of Tables. Since the characteristic of the logarithm of a number may be found by inspection, a table of logarithms contains only the mantissas. To make practical use of logarithms in computation it is necessary to have a conveniently arranged table from which we can find (*a*) the logarithm of any given number, and (*b*) the number corresponding to a given logarithm. Tables of logarithms differ according to the number of decimal places to which the mantissas are given and also in incidental details. However, the general principles governing their use are the same. These principles are explained for a four-place table (p. 536) by the following examples.

Problem *A*. ***To find the logarithm of a given number.***

(1) *When the number contains three or fewer figures.*

Example. To find the logarithm of 42.7.

First, by § 153, the characteristic is 1. We accordingly write (provisionally)

$$\log 42.7 = 1.$$

Next we look up in the tables the mantissa corresponding to the succession of figures 4, 2, 7. We run a finger down the first column of the table until we reach the figures 4, 2, hold it there while with another finger we mark the column headed with the third figure, 7. At the intersection of the line and column thus marked, we find the desired mantissa: 6304. The desired result is then

$$\log 42.7 = 1.6304.$$

To find the logarithm of 0.0427, we should proceed in precisely the same way, the only difference being that the characteristic is now 8 – 10. Hence,

$$\log 0.0427 = 8.6304 - 10.$$

(2) *When the number contains four significant figures.*

Example. To find log 32.73.

We see that again the characteristic is 1, and we write provisionally

$$\log 32.73 = 1.$$

Now, the mantissa of log 32.73 lies between the mantissas of log 32.70 and log 32.80; *i.e.* (from the table) between 5145 and 5159. The difference between these two mantissas (called the ***tabular difference*** at that place in the table) is 14, and this difference corresponds to a difference in the numbers of .10. According to the principle of linear interpolation,* the difference in the mantissas corresponding to a difference in the numbers of .03 is $14 \times .3 = 4.2$ or (rounded) 4. The mantissa corresponding to 3273 is then $5145 + 4 = 5149$, and we obtain

$$\log 32.73 = 1.5149.$$

Problem *B*. ***To find the number corresponding to a given logarithm.*** Here we simply reverse the preceding process.

Example. To find the number whose logarithm is 0.8485.

We first seek the mantissa 8485 in the table. We find that it lies between 8482 and 8488, corresponding respectively to the successions of figures 7050 and 7060. The tabular difference here is 6, while *our* difference, *i.e.* the difference we have to account for (8485 – 8482) is 3. Hence the corresponding difference in the numbers is $\frac{3}{6}$ of 10 or 5. Hence the succession of figures in the number sought is 7055. Since the characteristic is 0, the number sought is 7.055. Or, $\log 7.055 = 0.8485$.

If the mantissa is found exactly in the table, of course no interpolation is necessary. Thus the number whose logarithm is 9.7348 – 10 is 0.5430.

EXERCISES

1. Find the logarithms of the following numbers from the table on pp. 536–7: 482, 26.4, 6.857, 9001, 0.5932, 0.08628, 0.00038.

2. Find the numbers corresponding to the following logarithms: 2.7935, 0.3502, 7.9699 – 10, 9.5300 – 10, 3.6598, 1.0958.

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* One should convince oneself that the conditions for linear interpolation are satisfied by this table. In fact, it is readily seen that for several numbers immediately preceding and following 327, the tabular differences are 13 and 14.

155. Use of Logarithms in Computation. The way in which logarithms may be used in computation will be sufficiently explained in the following examples. A few devices often necessary or at least desirable will be introduced. The latter are usually self-explanatory. Reference is made to them here, in order that one may be sure to note them when they arise. The use of logarithms in computation depends, of course, on the fundamental properties derived in § 151.

Example 1. Find the value of 73.26 × 8.914 × 0.9214.

We find the logarithms of the factors, add them, and then find the number corresponding to this logarithm. The work may be arranged as follows:

Numbers		Logarithms
73.26	(→)	1.8649
8.914	(→)	0.9501
0.9214	(→)	9.9645 − 10
		12.7795 − 10
Product = 601.9 *Ans.*	(←)	2.7795

Example 2. Find the value of 732.6 ÷ 89.14.

Numbers		Logarithms
732.6	(→)	2.8649
89.14	(→)	1.9501
Quotient = 8.219 *Ans.*	(←)	0.9148

Example 3. Find the value of 89.14 ÷ 732.6.

Numbers		Logarithms
89.14	(→)	11.9501 − 10
732.6	(→)	2.8649
Quotient = 0.1217 *Ans.*	(←)	9.0852 − 10

Example 4. Find the value of $\frac{763.2 \times 21.63}{986.7}$.

Whenever an example involves several different operations on the logarithms as in this case, it is desirable to make out a *blank form*. When a blank form is used, all logarithms should be looked up first and entered in their proper places. After this has been done, the necessary operations (addition, subtraction, etc.) are performed. Such a procedure saves time and minimizes the chance of error.

FORM

Numbers		Logarithms
763.2	(→)	. . . . . .
21.63	(→)	(+) . . . . . .
product		. . . . . .
986.7	(→)	(−) . . . . . .
. . . . . *Ans.*	(←)	. . . . . .

FORM FILLED IN

Numbers		Logarithms
763.2	(→)	2.8826
21.63	(→)	1.3351
product		4.2177
986.7	(→)	2.9942
16.73 *Ans.*	(←)	1.2235

EXAMPLE 5. Find $(1.357)^5$.

Numbers		Logarithms
1.357	(→)	0.1326
$(1.357)^5 = 4.602$ *Ans.*	(←)	0.6630

EXAMPLE 6. Find the cube root of 30.11.

Numbers		Logarithms
30.11	(→)	1.4787
$\sqrt[3]{30.11} = 3.111$ *Ans.*	(←)	0.4929

EXAMPLE 7. Find the cube root of 0.08244.

Numbers		Logarithms
0.08244	(→)	28.9161 − 30
$\sqrt[3]{0.08244} = 0.4352$ *Ans.*	(←)	9.6387 − 10

EXERCISES

Compute the value of each of the following expressions using the table on pp. 536–537.

1. 34.96×4.65.
2. $518.7 \times 9.02 \times .0472$.
3. $\dfrac{0.5683}{0.3216}$.
4. $\dfrac{5.007 \times 2.483}{6.524 \times 1.110}$.
5. $(34.16 \times .238)^2$.
6. $8.572 \times 1.973 \times (.8723)^2$.
7. $\sqrt[3]{\dfrac{648.8}{(21.4)^2}}$.
8. $\sqrt{\dfrac{1379}{2791}}$.

9. $\sqrt{\frac{2.8076 \times 3.184}{(2.012)^3}}$.

10. $\sqrt[3]{\frac{2941 \times 17.32}{2173 \times 18.75}}$.

11. $\frac{\sqrt[3]{0.00732}}{\sqrt{735}}$.

12. $(20.027)^{\frac{1}{4}}$.

13. 2^{100}.

14. $\sqrt[3]{150^2 - 100^2}$.

15. $(0.02735)^{\frac{1}{5}}$.

16. $\frac{\sqrt{3275}}{(2.01)^{\frac{1}{3}}}$.

17. The stretch s of a brass wire when a weight m is hung at its free end is given by the formula

$$s = \frac{mgl}{\pi r^2 k},$$

where m is the weight applied in grams, $g = 980$, l is the length of the wire in centimeters, r is the radius of the wire in centimeters, and k is a constant. If $m = 844.9$ grams, $l = 200.9$ centimeters, $r = 0.30$ centimeters when $s = 0.056$, find k.

18. The crushing weight P in pounds of a wrought iron column is given by the formula

$$P = 299{,}600 \frac{d^{3.55}}{l^2},$$

where d is the diameter in inches and l is the length in feet. What weight will crush a wrought iron column 10 feet long and 2.7 inches in diameter?

19. The number n of vibrations per second made by a stretched string is given by the relation

$$n = \frac{1}{2\,l}\sqrt{\frac{Mg}{m}},$$

where l is the length of the string in centimeters, M is the weight in grams that stretches the string, m the weight in grams of one centimeter of the string, and $g = 980$. Find n when $M = 5467.9$ grams, $l = 78.5$ centimeters, $m = 0.0065$ grams.

20. The time t of oscillation of a pendulum of length l centimeters is given by the formula

$$t = \pi\sqrt{\frac{l}{980}}.$$

Find the time of oscillation of a pendulum 73.27 centimeters in length.

21. The weight w in grams of a cubic meter of aqueous vapor saturated at 17° C. is given by the formula

$$w = \frac{1293 \times 12.7 \times 5}{(1 + \frac{17}{273})(760 \times 8)}.$$

Compute w.

156. Exponential Equations. An equation in which the unknown is contained in an exponent is known as an *exponential equation.* Some such equations may be solved by taking the logarithms of both sides after the equation has been properly transformed.

Example 1. Solve the equation $3^{2x+1} + 7 = 15$.

Transposing the 7 and taking logarithms of both sides we obtain

$$(2x+1)\log 3 = \log 8.$$

Hence we find

$$x = \frac{1}{2}\left[\frac{\log 8}{\log 3} - 1\right].$$

Example 2. Money is placed at interest, compounded annually. Find a formula for the amount at the end of n years. Also a formula giving the number of years necessary to produce a given amount.

Let C be the original capital and r the given rate of interest (*i.e.* if the interest is 5 per cent, $r = 0.05$). The amount A_1 at the end of the first year is

$$A_1 = C + Cr = C(1+r).$$

At the end of two years we have

$$A_2 = A_1(1+r) = C(1+r)^2.$$

At the end of n years, the amount is

$$A = A_n = C(1+r)^n.$$

This is the formula required. To find n, given A, C, r, we take the logarithms of both sides and find

$$\log\frac{A}{C} = n\log(1+r), \text{ or } n = \frac{\log A - \log C}{\log(1+r)}.$$

EXERCISES

1. Solve for x the equation $2^x = 5$.

2. Solve for y the equation $3^y + 2 = 9$.

3. Solve for x and y the simultaneous equations $3^{x+y} = 4$, $2^{x-y} = 3$.

4. Solve for x and y the simultaneous equations $2^{x+y} = 6^y$, $3^{x-1} = 2^{y+1}$.

5. Find the amount of $1000 in 25 years at 5 per cent compound interest, compounded annually.

6. Find the amount of \$500 in 10 years at 4 per cent compound interest, compounded semiannually.

7. In how many years will a sum of money double itself at 5 per cent interest compounded annually? semiannually?

8. A thermometer bulb at a temperature of 20° C. is exposed to the air for 15 seconds, in which time the temperature drops 4 degrees. If the law of cooling is given by the formula $\theta = \theta_0 e^{-bt}$, where θ is the final temperature, θ_0 the initial temperature, e the natural base of logarithms, and t the time in seconds, find the value of b.

MISCELLANEOUS EXERCISES

1. What objections are there to the use of a negative number as the base of a system of logarithms?

2. Show that $a^{\log_a x} = x$.

3. Write each of the following expressions as a single term:

(*a*) $\log x + \log y - \log z$. (*b*) $3 \log x - 2 \log y + 3 \log z$.

(*c*) $3 \log a - \log (x + y) - \frac{1}{2} \log (cx + d) + \log \sqrt{w + x}$.

4. Solve for x the following equations:

(*a*) $2 \log_2 x + \log_2 4 = 1$. (*c*) $2 \log_{10} x - 3 \log_{10} 2 = 4$.

(*b*) $\log_3 x - 3 \log_3 2 = 4$. (*d*) $3 \log_2 x + 2 \log_2 3 = 1$.

5. How many digits are there in 2^{85}? 3^{142}? $3^{12} \times 2^8$?

6. Which is the greater, $(\frac{21}{20})^{100}$ or 100?

7. Find the value of each of the following expressions. (See § 152.)

(*a*) $\log_6 35$. (*b*) $\log_3 34$. (*c*) $\log_7 245$. (*d*) $\log_{13} 26$.

8. Prove that $\log_b a \cdot \log_a b = 1$.

9. Prove that

$$\log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a [x + \sqrt{x^2 - 1}].$$

10. The velocity v in feet per second of a body that has fallen s feet is given by the formula $v = \sqrt{64.3\, s}$.

What is the velocity acquired by the body if it falls 45 ft. 7 in.?

11. Solve for x and y the equations: $2^x = 16^y$, $x + 4y = 4$.

CHAPTER IX

NUMERICAL COMPUTATION

I. ERRORS IN COMPUTATION

157. Absolute and Relative Errors. In § 29 we noted that the numerical result of every observed measurement is an approximation. The difference between the exact value of the magnitude and this observed value is a concrete number called the ***absolute error***.* Often the absolute error is not the most serviceable measure of the precision of a measurement. The ***relative error***, which is defined as the ratio of the absolute error to the exact value, is often found more serviceable. Since the relative error is a ratio, it is an abstract number, and is therefore sometimes expressed in per cent. For example, if the diagonal of a square 10 in. on a side be measured and and found to be 14.1 in., the absolute error is less than 1/10 of an inch. The relative error is less than $(1/10) \div 10\sqrt{2} = 1/141$, approximately, *i.e.* less than 0.71 per cent.

158. Rounded Numbers. Significant Figures. When the result of a measurement is expressed in the decimal notation, a generally adopted convention makes it possible to determine the degree of precision of the measurement from the number of significant figures contained in the number expressing the measure. This convention simply specifies that no more digits shall be written than are (probably) correct. Thus a measurement of a length expressed as 14.1 in. means that the measure is exact to the *nearest tenth of an inch.* If on the other hand the measurement of this length were exact to the nearest hundredth of an inch, the measure would have been expressed by the number 14.10.*

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* The absolute error is therefore positive or negative according as the observed value is too small or too large.

We should note, then, that the two numbers 14.1 and 14.10 do not mean precisely the same thing, when they express the result of a measurement.

Again we may note that the absolute errors involved in the expression 4371.52 ft. and 42.81 ft. are each less than one hundredth of a foot; whereas the relative error is in the first case less than 1/437152 and in the second only less than 1/4281.

Sometimes we are furnished with numbers expressing measures which are given with greater accuracy than we can use, or care to use. Thus suppose we want to express a measured length of 3.5 in. in terms of centimeters. We find in a table of equivalent lengths that 1 in. = 2.54001 cm. It would be obviously absurd to use this expression as it stands. We accordingly *round it off* to 2.54 or even to 2.5 and find that 3.5 in. = 8.9 cm. If, on the other hand, we wish to express 3.50000 in. in centimeters, we should have to use the value 2.54001.

A number is rounded off by dropping one or more digits at the right, and, if the last digit dropped is 5^+, 6, 7, 8, or 9, increasing the preceding digit by 1.† Thus the successive approximations to π obtained by rounding off $3.14159 \cdots$ are 3.1416, 3.142, 3.14, 3.1, 3.

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* In other words $x = 14.1$ means that the exact value of x lies *between* 14.05 and 14.15; and $x = 14.10$ means that the exact value of x lies *between* 14.095 and 14.105.

† In rounding off a 5, computers use the following rule: Always round off a 5 to an *even* digit. Thus 1.415 would be rounded to 1.42, whereas 1.445 would be rounded to 1.44. The reason for this rule is that, if used consistently, the errors made will in the long run compensate each other.

The ***significant figures*** of a number may now be defined as the digits 1, 2, 3, ⋯, 9 together with such zeros as occur between them or as have been properly retained in rounding them off. Thus 34.96 and 3,496,000 are both numbers of four significant figures. On the other hand 3,496,000.0 has eight significant figures, since the 0 in the first decimal place according to the convention adopted means that the number is exact to the nearest tenth. This zero is then essentially a digit properly retained in rounding off, and should be counted as one of the significant figures.

Confusion can arise in only one case. For example, if the number 3999.7 were rounded by dropping the 7, we should write it as 4000 which, according to the rule just given, we would consider as having only one significant figure, whereas in reality we know from the way in which the number was obtained that all four of the figures are significant. When such a case arises in practice we may simply remember the fact, or we can indicate that the zeros are significant by underscoring them, or by some other device. Computers adopt devices of their own to avoid errors in such cases.

159. Computation with Rounded Numbers. Addition. Since the (absolute) error of any approximate number can be at most one half the unit represented by the last digit at the right, the sum of n such numbers can be in error by at most $n/2$ times the unit represented by the last figure. These considerations lead to the following convention: in adding a column of approximate numbers first round off the given numbers so that not more than one column at the right is broken; round off the sum so that the last figure to the right comes in the last unbroken column. This last figure is then uncertain. Nevertheless, it is usually retained temporarily. As a matter of fact, even the figure preceding this last one is not certain, since the errors may accumulate in adding several numbers.

For example, in adding

$$\begin{array}{r} 21.64 \\ 3.8576 \\ 5.259743 \\ 10.31 \end{array}$$

we first round off:

$$\begin{array}{l} 21.64 \\ 3.858 \\ 5.260 \\ \underline{10.31} \\ 41.068 = 41.07 \end{array}$$

The final sum is written 41.07, but even the last figure 7 is open to question. Show that the true result may be as low as 41.06 or as high as 41.08.

To retain all the figures in the second and third of the numbers originally given would be absurd and would give in the result a misleading *pretense* of accuracy which does not exist in fact.

In subtraction round off similarly.

160. Multiplication. Let a and b be approximate numbers and let their relative errors be α and β respectively. The exact numbers are then (nearly) $a + a\alpha$ and $b + b\beta$. Their product is

$$ab + ab\alpha + ab\beta + ab\alpha\beta.$$

The error committed in using ab as the product is then

$$ab(\alpha + \beta + \alpha\beta)$$

and the relative error is therefore nearly

$$\alpha + \beta + \alpha\beta.$$

Now in practice α and β are small fractions, so that $\alpha\beta$ is insignificant when compared with $\alpha + \beta$. (For example, if α and

β are both equal approximately to 0.001, $\alpha\beta$ is equal approximately to 0.000001.) Hence, we conclude that *the relative error of the product of two numbers is approximately equal to the* **sum** *of the relative errors of the factors.*

Hence, in finding the product of two approximate numbers, round off so that the two numbers have the same number of *significant figures,* and *retain only this number of significant figures in the product.** Even then the last figure retained may be unreliable.

Example. Multiply 27.17 by 3.14159. Round off the second factor to 3.142, and multiply:

$$\begin{array}{r}
\underline{27.17 \times 3.142} \\
5434 \\
10868 \\
2717 \\
\underline{8151} \\
85.36814 = 85.37
\end{array}$$

Even the figure 7 may be in error. Show that the true answer may be as low as 85.35.

The labor involved in such a multiplication may be considerably reduced by slightly modifying the method used, as follows:

$$\begin{array}{r}
\underline{27.170 \times 3.142} \\
81510 \\
2717 \\
1087 \\
\underline{54} \\
85368 = 85.37
\end{array}$$

After having equalized the number of significant figures annex a zero to the multiplicand. Multiply by the first figure on the left of the multiplier. Drop the last figure of the multiplicand and multiply by the second figure of the multiplier. Drop the next figure of the multiplicand and multiply by the third figure of the multiplier (but "carry" the amount from the figure dropped: thus in the example having dropped the 7 and multiplying by 4, we say $4 \times 7 = 28$, carry 3, $4 \times 1 = 4$, $+ 3 = 7$, which is the first figure we write), and so on, arranging all the partial products so that the last figures from the left fall into the same vertical column; then add in the usual way.

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* Since $\frac{22}{7} = 3.1428571$, while $\pi = 3.1415927$, the value $\frac{22}{7}$ may be used for π when the uncertainty of the other factors in a product in which it appears is greater than 1 part in 3000 (approximately).

161. Division. In case either the dividend (N) or the divisor (D) is an approximate number, the following shortened method may be used:

1. Equalize the relative accuracy of N and D; but if D is larger at the left, keep one extra figure on N (as in the example below).

2. Divide as in long division, but drop successive figures in D, instead of adding successive zeros to N.

Example. Find 295.679 ÷ 7.53. (As 7 is greater than 2, we retain four figures in the dividend.)

```
7.53| 295.7 |39.3
      225 9  [3 × 753]
       69 8  [divide by 75, gives 9]
       67 8  [9 × 3 = 27, carry 3 ; 9 × 75 = 675, + 3 = 678]
        2 0  [divide by 7, gives 3 (nearer than 2)]
```

EXERCISES

1. Add the following numbers, each representing the result of a measurement: 25.62, 341.718, 2.62394, 28.7125.

2. Express 5.216 inches in centimeters.

3. Express 53.291 cm. in inches.

4. A rectangular table top is measured, and is found to be $2'4''.5 \times 3'6''.4$. Find its area. Find the error caused in this area if the measurements are each $0''.1$ too short. Find the relative error in the area.

5. Assuming that you can estimate the length and the breadth of a room which is about 15′ by 18′ to within 2′, how nearly can you estimate its area?

6. Assuming that you can measure each of the dimensions of the room of Ex. 5 with a yardstick to within 1″ error, how nearly can you find the area of the floor? If the height of the room is about 10′, how nearly can you find the volume of the room by measurement?

7. Assuming that you can measure the radius of a circle about 5″ in diameter to within $0''.1$ error, how nearly can you find its area? How nearly could you find by measurement the volume of a cylinder about 5′ high and about 5″ in diameter?

II. LOGARITHMIC SOLUTION OF TRIANGLES

162. Logarithmic Computation. We have already had occasion to observe that many computations in engineering, astronomy, etc., are carried out by means of logarithms. In the last chapter a few examples of the use of logarithms in computation were given in connection with a four-place table. Such a table suffices for data and results accurate to four significant figures. When greater accuracy is desired we use a five-, six-, or seven-place table.

The methods used in connection with such a table differ slightly from those used ordinarily with a four-place table. Accordingly we take up briefly at this point some problems involving computation with a five-place table of logarithms.

No subject is better adapted to illustrate the use of logarithmic computation than the solution of triangles, which we shall consider in some detail. Five-place tables and logarithmic solutions ordinarily are used at the same time, since both tend toward greater speed and accuracy.

163. Five-place Tables of Logarithms and Trigonometric Functions. The use of a five-place table of logarithms differs from that of a four-place table in the general use of so-called "interpolation tables" or "tables of proportional parts," to facilitate interpolation. Since the use of such tables of proportional parts is fully explained in every good set of tables, it is unnecessary to give such an explanation here. It will be assumed that the student has made himself familiar with their use.*

In the logarithmic solution of a triangle we nearly always need to find the logarithms of certain trigonometric functions.

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* For this chapter, such a five-place table should be purchased. See, for example, THE MACMILLAN TABLES, which contain all the tables mentioned here with an explanation of their use.

For example, if the angles A and B and the side a are given, we find the side b from the law of sines given in § 125,

$$b = \frac{a \sin B}{\sin A}.$$

To use logarithms we should then have to find $\log a$, $\log (\sin B)$ and $\log (\sin A)$. With only a table of natural functions and a table of logarithms at our disposal, we should have to find first $\sin A$, and then $\log \sin A$. For example, if $A = 36° 20'$, we would find $\sin 36° 20' = 0.59248$, and from this would find $\log \sin 36° 20' = \log 0.59248 = 9.77268 - 10$. This double use of tables has been made unnecessary by the direct tabulation of the logarithms of the trigonometric functions in terms of the angles. Such tables are called tables of logarithmic sines, logarithmic cosines, etc. Their use is explained in any good set of tables.

The following exercises are for the purpose of familiarizing the student with the use of such tables.

EXERCISES

1. Find the following logarithms: *

(*a*) $\log \cos 27° 40'.5$.
(*b*) $\log \tan 85° 20'.2$.
(*c*) $\log \sin 45° 40'.7$.
(*d*) $\log \operatorname{ctn} 86° 53'.6$.
(*e*) $\log \cos 87° 6'.2$.
(*f*) $\log \cos 36° 53'.3$.

2. Find A, when

(*a*) $\log \sin A = 9.81632 - 10$.
(*b*) $\log \cos A = 9.97970 - 10$.
(*c*) $\log \tan A = 0.45704$.
(*d*) $\log \sin A = 9.78332 - 10$.
(*e*) $\log \operatorname{ctn} \frac{1}{2} A = 0.70352$.
(*f*) $\log \tan \frac{1}{2} A = 9.94365 - 10$.

3. Find θ, if $\tan \theta = \dfrac{476.32 \times 89.710}{87325}$.

4. Given a triangle ABC, in which $\angle A = 32°$, $\angle B = 27°$, $a = 5.2$, find b by use of logarithms.

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* Five-place logarithms are properly used when angles are measured to the nearest tenth of a minute. For accuracy to the nearest second, six places should be used.

164. The Logarithmic Solution of Triangles. The effective use of logarithms in numerical computation depends largely on a proper arrangement of the work. In order to secure this, the arrangement should be carefully planned beforehand by constructing a *blank form,* which is afterwards filled in. Moreover a practical computation is not complete until its accuracy has been checked. The blank form should provide also for a good check. Most computers find it advantageous to arrange the work in two columns, the one at the left containing the *given numbers* and the *computed results,* the one on the right containing the logarithms of the numbers each in the same horizontal line with its number. The work should be so arranged that every number or logarithm that appears is properly labeled; for it often happens that the same number or logarithm is used several times in the same computation and it should be possible to locate it at a glance when it is wanted.

The solution of triangles may be conveniently classified under four cases:

CASE I. *Given two angles and one side.*

CASE II. *Given two sides and the angle opposite one of the sides.*

CASE III. *Given two sides and the included angle.*

CASE IV. *Given the three sides.*

In each case it is desirable (1) to draw a figure representing the triangle to be solved with sufficient accuracy to serve as a rough check on the results; (2) to write out all the formulas needed for the solution and the check; (3) to prepare a blank form for the logarithmic solution on the basis of these formulas; (4) to fill in the blank form and thus to complete the solution.

We give a sample of a blank form under Case I; the student should prepare his own forms for the other cases.

165. Case I. Given two Angles and one Side.

Example. *Given*: $a = 430.17$, $A = 47° 13'.2$, $B = 52° 29'.6$. (Fig. 134)
To find: C, b, c.
Formulas:

$$C = 180° - (A + B),$$

$$b = \frac{a}{\sin A} \sin B,$$

$$c = \frac{a}{\sin A} \sin C.$$

Check: $$\frac{c - b}{c + b} = \frac{\tan \frac{1}{2}(C - B)}{\tan \frac{1}{2}(C + B)}.$$

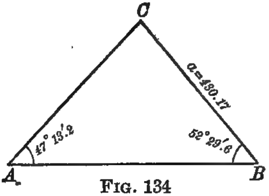

Fig. 134

The following is a convenient *blank form* for the logarithmic solution. The sign (+) indicates that the numbers should be added; the sign (−) indicates that the number should be subtracted from the one just above it.

Numbers Logarithms

$A = $ . . . . .
(+) $B = $ . . . . .
$A + B = $ . . . . .
179° 60'.0
$C = $ . . . . .
$a = $ . . . . . (→) . . . . . .
$\sin A = \sin$ . . . (→) (−) . . . . . .

$a/\sin A$ . . . . . .
$\sin B = \sin$ . . . (→) (+) . . . . . .
$b = $ . . . . . (←) . . . . . .

$a/\sin A$ . . . . . .
$\sin C = \sin$ . . . (→) (+) . . . . . .
$c = $ . . . . . (←) . . . . . .

Check

$c - b = $ . . . . . (→) . . . . . .
$c + b = $ . . . . . (→) (−) . . . . . .
(1)
$C - B = $ . . . . . .
$C + B = $ . . . . . .
$\tan \frac{1}{2}(C - B) = \tan$ . . . (→) . . . . . .
$\tan \frac{1}{2}(C + B) = \tan$ . . . (→) (−) . . . . . .
(2)

(Logs (1) and (2) should be equal for check.)

Filling in this blank form, we obtain the solution as follows.

Numbers			Logarithms
$A = 47° 13'.2$			
$B = 52° 29'.6$			
$A + B = 99° 42'.8$			
$179° 60'.0$			
$C = 80° 17'.2$			
$a = 430.17$	$(\rightarrow)$		2.63364
$\sin A = \sin 47° 13'.2$	$(\rightarrow)$	$(-)$	9.86567 − 10
$a/\sin A$			2.76797
$\sin B = \sin 52° 29'.6$	$(\rightarrow)$	$(+)$	9.89943 − 10
$b = 464.94$ *Ans.*	$(\leftarrow)$		2.66740
$a/\sin A$			2.76797
$\sin C = \sin 80° 17'.2$	$(\rightarrow)$	$(+)$	9.99373
$c = 577.70$ *Ans.*	$(\leftarrow)$		2.76170

Check

$c - b = 112.76$	$(\rightarrow)$		2.05215	
$c + b = 1042.64$	$(\rightarrow)$	$(-)$	3.01813	
			9.03402 − 10	*Check*
$C - B = 27° 47'.6$				
$C + B = 132° 46'.8$				
$\tan \frac{1}{2}(C - B) = \tan 13° 53'.8$	$(\rightarrow)$		9.39342 − 10	
$\tan \frac{1}{2}(C + B) = \tan 66° 23'.4$	$(\rightarrow)$	$(-)$	0.35942	
			9.03400 − 10	

EXERCISES

Solve and check the following triangles ABC:

1. $a = 372.5, \ A = 25° 30', \ B = 47° 50'.$

2. $c = 327.85, \ A = 110° 52'.9, \ B = 40° 31'.7.$ *Ans.* $C = 28° 35'.4$
$a = 640.11, \ b = 445.20.$

3. $a = 53.276, \ A = 108° 50'.0, \ C = 57° 13'.2.$

4. $b = 22.766, \ B = 141° 59'.1, \ C = 25° 12'.4.$

5. $b = 1000.0, \ B = 30° 30'.5, \ C = 50° 50'.8.$

6. $a = 257.7, \ A = 47° 25', \ B = 32° 26'.$

166. Case II. Given two Sides and an Angle opposite one of them.

If A, a, b are given, B may be determined from the relation

$$\sin B = \frac{b \sin A}{a}. \tag{1}$$

If $\log \sin B = 0$, the triangle is a right triangle. Why?

If $\log \sin B > 0$, the triangle is impossible. Why?

If $\log \sin B < 0$, there are two possible values B_1, B_2 of B, which are supplementary.

Hence there may be two solutions of the triangle. (See Ex. 1, page 249.)

No confusion need arise from the various possibilities if the corresponding figure is constructed and kept in mind.

It is desirable to go through the computation for $\log \sin B$ before making out the rest of the blank form, unless the data obviously show what the conditions of the problem actually are.

EXAMPLE 1. *Given:* $A = 46° 22'.2$, $a = 1.4063$, $b = 2.1048$. (Fig. 135)
To find: B, C, c.
Formula: $\sin B = \dfrac{b \sin A}{a}$.

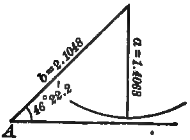

FIG. 135

	Numbers			Logarithms
b	$= 2.1048$	$(\rightarrow)$		0.32321
$\sin A$	$= \sin 46° 22'.2$	$(\rightarrow)$	$(+)$	$9.85962 - 10$
$b \sin A$				0.18283
a	$= 1.4063$	$(\rightarrow)$	$(-)$	0.14808
$\sin B$		$(\leftarrow)$		0.03475

Hence the triangle is impossible. Why?

Example 2. *Given:* $a = 73.221$, $b = 101.53$, $A = 40° 22'.3$. (Fig. 136)
To find: B, C, c.
Formula: $\sin B = \dfrac{b \sin A}{a}$.

Numbers			Logarithms
$b = 101.53$	$(\rightarrow)$		2.00660
$\sin A = \sin 40° 22'.3$	$(\rightarrow)$	$(+)$	9.81140 − 10
$b \sin A$			11.81800 − 10
$a = 73.221$	$(\rightarrow)$	$(-)$	1.86464
$\sin B$			9.95336 − 10

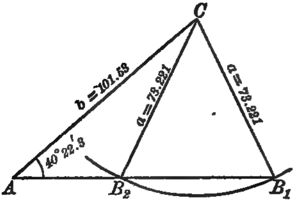

Fig. 136

The triangle is therefore possible and has two solutions (as the figure shows). We then proceed with the solution as follows:

We find one value B_1 of B from the value of log sin B. The other value B_2 of B is then given by $B_2 = 180° - B_1$.

Other formulas:

$$C = 180° - (A + B).$$

$$c = \frac{a \sin C}{\sin A}.$$

$$\textit{Check:}\ \frac{c-b}{c+b} = \frac{\tan \frac{1}{2}(C-B)}{\tan \frac{1}{2}(C+B)}.$$

Numbers				Logarithms
$\sin B$				9.95336 − 10
$B_1 =$	63° 55′.2			
	179° 60′.0			
$B_2 =$	116° 4′.8			
$A + B_1 =$	104° 17′.5			
	179° 60′.0			
$C_1 =$	75° 42′.5			
a		$(\rightarrow)$		1.86464
$\sin A$		$(\rightarrow)$	$(-)$	9.81140 − 10
$a/\sin A$				2.05324
$\sin C_1 =$	$\sin 75° 42.'5$	$(\rightarrow)$	$(+)$	9.98634 − 10
$c_1 =$	109.54	$(\leftarrow)$		2.03958

$$\left.\begin{array}{rlll}
c_1 - b = & 8.01 & (\rightarrow) & 0.90363 \\
c_1 + b = & 211.07 & (\rightarrow)\ (-) & \underline{2.32443} \\
 & & & 8.57920 - 10 \\
C_1 - B_1 = & 11^\circ\, 47'.3 & & \\
C_1 + B_1 = & 139^\circ\, 37'.7 & & \\
\tan \tfrac{1}{2}(C_1 - B_1) = & \tan 5^\circ\, 53'.6 & (\rightarrow) & 9.01377 - 10 \\
\tan \tfrac{1}{2}(C_1 + B_1) = & \tan 69^\circ\, 48'.8 & (\rightarrow) & \underline{0.43455} \\
 & & & 8.57922 - 10
\end{array}\right\}\ \textit{Check}\,^*$$

One solution of the triangle gives, therefore, $B = 63^\circ\, 55'.2$, $C = 75^\circ\, 42'.5$, $c = 109.54$.

To obtain the second solution, we begin with $B_2 = 116^\circ\, 4'.8$. We find C_2 from $C_2 = 180^\circ - (A + B_2)$; *i.e.* $C_2 = 23^\circ\, 32'.9$. The rest of the computation is similar to that above and is left as an exercise.

EXERCISES

1. Show that, given A, a, b, if A is obtuse, or if A is acute and $a > b$, there cannot be more than one solution.

Solve the following triangles and check the solutions:

2. $a = 32.479$, $b = 40.176$, $A = 37^\circ\, 25'.1$.

3. $b = 4168.2$, $c = 3179.8$, $B = 51^\circ\, 21'.4$.

4. $a = 2.4621$, $b = 4.1347$, $B = 101^\circ\, 37'.3$.

5. $a = 421.6$, $c = 532.7$, $A = 49^\circ\, 21'.8$.

6. $a = 461.5$, $c = 121.2$, $C = 22^\circ\, 31'.6$.

7. Find the areas of the triangles in Exs. 2–5.

167. Case III. Given two Sides and the Included Angle.

EXAMPLE. *Given:* $a = 214.17$, $b = 356.21$, $C = 62^\circ\, 21'.4$. (Fig. 137)

To find: A, B, c.

Formulas:

$$\tan \tfrac{1}{2}(B - A) = \frac{b - a}{b + a} \tan \tfrac{1}{2}(B + A);$$

$$B + A = 180^\circ - C = 117^\circ\, 38'.6;$$

$$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B}.$$

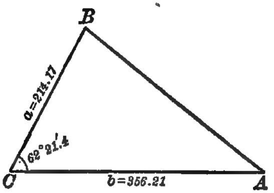

FIG. 137

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* A small discrepancy in the last figure need not cause concern. Why?

Numbers		Logarithms
$b - a = 142.04$	(→)	2.15241
$b + a = 570.38$	(→)	(−) 2.75616
$(b - a)/(b + a)$		9.39625 − 10
$\tan \frac{1}{2}(B + A) = \tan 58° 49'.3$	(→)	(+) 0.21817
$\tan \frac{1}{2}(B - A) = \tan 22° 22'.2$	(←)	9.61442 − 10
$\therefore A = 36° 27'.1$ *Ans.*		
$B = 81° 11'.5$ *Ans.*		
$a = 214.17$	(→)	2.33076
$\sin A = \sin 36° 27'.1$	(→)	(−) 9.77389 − 10
$a/\sin A$		2.55687
$\sin C = \sin 62° 21'.4$	(→)	(+) 9.94736 − 10
$c = 319.32$ *Ans.*	(←)	2.50423

Check by finding log $(b/\sin B)$.

EXERCISES

Solve and check each of the following triangles.

1. $a = 74.801$, $b = 37.502$, $C = 63° 35'.5$.

2. $a = 423.84$, $b = 350.11$, $C = 43° 14'.7$.

3. $b = 275$, $c = 315$, $A = 30° 30'$.

4. $a = 150.17$, $c = 251.09$, $B = 40° 40'.2$.

5. $a = 0.25089$, $b = 0.30007$, $C = 42° 30' 20''$.

6. Find the areas of the triangles in Exs. 1–5.

168. Case IV. Given the three Sides.

EXAMPLE. *Given:* $a = 261.62$,
$b = 322.42$,
$c = 291.48$.
To find: A, B, C.

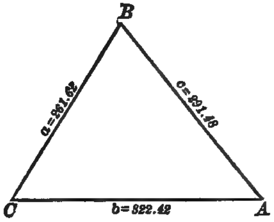

FIG. 138

Formulas:

$$s = \tfrac{1}{2}(a + b + c).$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c}. \qquad (\S\ 143)$$

Check: $A + B + C = 180°$.

Numbers			Logarithms
$a = 261.62$			
$b = 322.42$			
$c = 291.48$			
$2s = 875.52$			
$s = 437.76$			
$s - a = 176.14$	$(\rightarrow)$		2.24586
$s - b = 115.34$	$(\rightarrow)$		2.06198
$s - c = 146.28$	$(\rightarrow)$	$(+)$	2.16518
$2s = 875.52$	*(Check.)*		6.47302
$s = 437.76$	$(\rightarrow)$	$(-)$	2.64124
r^2			3.83178
r			1.91589
$s - a$			2.24586
$\tan \frac{1}{2} A = \tan 25^\circ\ 4'.1$	$(\leftarrow)$		$9.67003 - 10$
r			1.91589
$s - b$			2.06198
$\tan \frac{1}{2} B = \tan 35^\circ\ 32'.4$	$(\leftarrow)$		$9.85391 - 10$
$r =$			1.91589
$s - c =$			2.16518
$\tan \frac{1}{2} C = \tan 29^\circ\ 23'.4+$	$(\leftarrow)$		$9.75071 - 10$
$A = 50^\circ\ 8'.2$	*Ans.*		
$B = 71^\circ\ 4'.8$	*Ans.*		
$C = 58^\circ\ 46'.9$	*Ans.*		
$179^\circ\ 59'.9$	*Check.*		

EXERCISES

Solve and check each of the following triangles:

1. $a = 2.4169,\ b = 3.2417,\ c = 4.6293.$

2. $a = 21.637,\ b = 10.429,\ c = 14.221.$

3. $a = 528.62,\ b = 499.82,\ c = 321.77.$

4. $a = 2179.1,\ b = 3467.0,\ c = 5061.8.$

5. $a = 0.1214,\ b = 0.0961,\ c = 0.1573.$

6. Find the areas of the triangles in Exs. 1–5.

7. Find the areas of the inscribed circles of the triangles in Ex. 1–5.

III. THE LOGARITHMIC SCALE — THE SLIDE RULE

169. The Logarithmic Scale. Let us lay off, on a straight line, segments issuing from the same origin and proportional to the logarithms of the numbers 1, 2, 3, 4, …. The base of the system of logarithms is immaterial. Let us label the end-points of these segments by the corresponding numbers. This gives a *non-uniform* scale, which is called a ***logarithmic scale.*** Such a scale is pictured in Fig. 139.

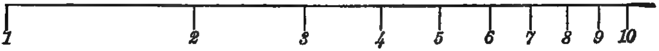

FIG. 139

A scale of this kind is easily constructed from the graph of the logarithmic function (Fig. 133).

170. The Slide Rule. The slide rule is an instrument often used by engineers and others who do much computing.* It consists of a rule (usually made of wood faced with celluloid) along the center of which a slip of the same material slides in a groove. This slip is called the ***slide.*** The face of the slide is level with the face of the rule.

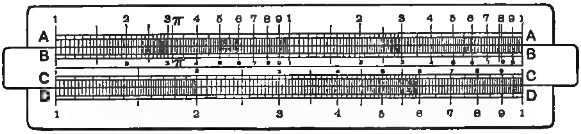

FIG. 140

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* Engineers usually purchase rather expensive slide rules made of wood and celluloid. These are on sale in all stores which carry draftsmen's supplies. A very simple slide rule sufficiently accurate for class purposes is printed on hard pasteboard and is obtainable at reasonably small cost through any one of several manufacturers of instruments. Figure 140 is reproduced on a larger scale on the first fly-leaf at the back of the book. By cutting out this leaf and carefully cutting up the figure, a slide rule can be made by the student. This will not be very accurate, but it will suffice to illustrate the principles.

Along the upper edge of the groove are engraved two logarithmic scales, usually labeled *A* and *B*, the scale *A* being on the rule, the scale *B* on the slide. (See Fig. 141.)

The scales *A* and *B* are identical. The slide is simply a mechanical device for adding graphically the segments on these scales. Since the segments represent the logarithms of the numbers found on the scale, the operation of adding the segments is equivalent to multiplying the corresponding numbers. Thus, to find the product 2.5 × 3.2 move the slide to the right until the point marked 1 at the extreme left of the slide (scale *B*) is in contact with the point 2.5 on scale *A* (Fig. 141 shows the positions of scales *A* and *B* after this operation). The point 3.2 on scale *B* is then opposite the point 8.0 on scale *A*. The latter number is the required product: 2.5 × 3.2 = 8.0. A little reflection should make quite clear how the operation just performed is equivalent to adding the logarithms of 2.5 and 3.2 and then reading from the scale the number corresponding to the sum. We may note further that with slide set as in the example just worked it is set for multiplying any number by 2.5; *i.e.* every number of the scale *A* is the product of 2.5 by the number below it on scale *B*.

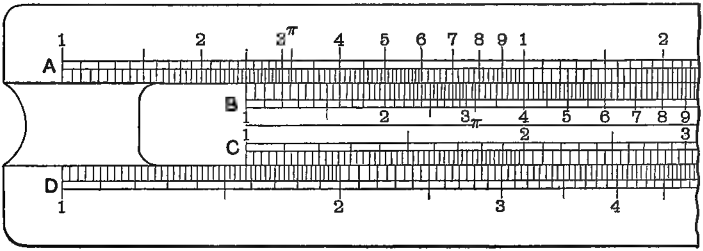

FIG. 141

The slide is therefore also set for division by 2.5. Every number of scale *B* is the result of dividing the number above it by 2.5. Thus we read from the scale (set as before) that $7.2 \div 2.5 = 2.9$ approximately.

Having now shown very briefly how the slide rule may be used for multiplication and division, let us examine it a little more closely. Scales *A* and *B* are labeled with the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, \cdots, 9, 1.$$

It is natural to ask why the number following the 9 in the middle of these scales is not labeled 10? The answer is that the numbers on the slide rule are given without any reference to the position of the decimal point, just as the numbers in a table of logarithms are given without reference to the decimal point. The number 1 at the extreme left of the scale may represent either 1, or 10, or 100, or 1000, etc., or .1, or .01, or .001, etc. If the 1 at the extreme left of the scale represents 1, then the other numbers on the first half of the scale represent 2, 3, ···, 9, the 1 in the middle represents 10, the 2 represents 20, and the successive numbers represent 30, 40, ···, 100 (the last being represented by the 1 at the extreme right of the scale). If on the other hand the 1 at the left represents 100, the successive numbers represent 200, 300, ···, 900, 1000, 2000, ···, 10,000. If the 1 at the left represents .1, the successive numbers represent .2, .3, ···, .9, 1.0, 2.0, ···, 10.0; and so on.

The reading of the subdivisions on the scales (*A* and *B*) should now offer little difficulty. Whenever an interval between two successive numbers is divided by certain lines of the same length into 10 parts, each of these parts represents one tenth of the number represented by the interval in question. Thus, if we fix our attention on the division between 2 and 3, we note that a certain set of lines divides this interval into 10 parts; if the 2 represent 2, these divisions represent respectively 2.1, 2.2, ···, 2.9. On the other hand, if the 2 is thought of as representing 20, these divisions represent 21, 22, ⋯, 29: and so on. These divisions into ten are at some parts of the scale subdivided further into five or two parts. These parts then represent fifths or halves of the interval that represented a tenth. Thus we may readily locate on the scale the point representing 1.42 or the point representing 3.65.

Turning our attention to scales C and D along the lower edge of the groove on the slide and the rule respectively, we note first that these two scales are also identical. Comparing them with scales A and B, we see that the unit chosen for C and D is just twice the unit of A and B. Hence the scales C and D can be used for multiplying and dividing just as scales A and B are used; however on C and D our range is smaller. The range of numbers on A and B is from 1 to 100; on C and D only from 1 to 10. To make up for this limitation, scales C and D give greater accuracy.

However, the principal reason for the existence of the second pair of scales is the fact that the two pairs of scales thus obtained furnish *a table of squares and square roots.* In view of the relation between the units with respect to which the two pairs of scales are constructed, every number of scale A is the square of the number vertically below it on scale D. Why? In order that corresponding numbers on scales A and D may be accurately read off, every slide rule is provided with a *runner*, the vertical line on which connects corresponding numbers of the upper and lower scales. The runner also enables us to perform calculations consisting of several operations without reading off the intermediate results, thus saving time and securing greater accuracy in the final result. The actual use of the slide rule will be explained in the next article.

The successful use of the slide rule depends largely on the ability to read the scales readily and accurately, accuracy often necessitating the estimating of numbers falling between the lines of division. The ability mentioned can be secured only by practice. A proficient operator, with a ten-inch slide rule, can always secure results accurate to three significant figures. This degree of accuracy is sufficient for many of the computations of applied science, manufacturing, etc., in which the slide rule is proving more and more useful.

171. The Use of the Slide Rule. All calculations in multiplication, division, proportion, etc., are worked on scales C and D unless the answer is so large that it does not lie on the scale. In that case scales A and B are used. Let us begin with *proportion.* On this topic, and on the corresponding property of the slide rule, all computations involving multiplication or division, or both, may be made to depend in a very simple way.

The property of the slide rule referred to is as follows: No matter where the slide be placed, *all the numbers on the slide bear the same ratio to the corresponding numbers on the rule* (due regard being had to the position of the decimal point). For example, if the slide be set so that 2 of C coincides with 4 of D, it will be observed that the same ratio $2:4$ exists between every pair of corresponding numbers: $1:2$, $3:6$, $42:84$, $125:250$, etc. Explain why this is true. This leads at once to the rule for finding the fourth term of a proportion, when the first three are given. We give this rule in diagrammatic form, as follows:*

To find the fourth term of a proportion:

C D	Set first term over second term.	Under third term find fourth term.

---

* In this article we have followed to a considerable extent the treatment given in the Manual for the use of the Mannheim Slide Rule, published by the Keuffel and Esser Co., New York.

This gives the solution of the equation

$$\frac{a}{b}=\frac{c}{x}.$$

To find the **product** *ab, solve the proportion*

$$\frac{1}{a}=\frac{b}{x}.$$

To find the **quotient** $\frac{a}{b}$, *solve the proportion*

$$\frac{a}{b}=\frac{x}{1}.$$

The following examples will make clear the procedure.

EXAMPLE 1. *Solve the proportion:* $13/24 = 32/x$.

C	Set 13	Under 32
D	over 24	find 59.1 *Ans.*

EXAMPLE 2. *Solve the proportion:* $13/24 = 75/x$.

Since the first two terms of the proportion are the same as in the preceding example, we set the slide as before. We now find, however, that 75 on *C* is beyond the extremity of *D*. We accordingly set the runner on the left-hand 1 of *C*, and then set the right-hand 1 of *C* on the runner. We find under 75 the number 138.5, the required value of x.* (Justify the above use of the runner.)

The same example can be done on scales *A* and *B* with one setting, without using the runner.

EXAMPLE 3. *Find the product:* 23.2×5.3.

C	Set 1	Under 5.3
D	over 23.2	find 123.0 *Ans.*

Here we set the right-hand 1 on 23.2. Use whichever 1 serves. The decimal point, in this as in the other examples, is simply located by inspection and a brief mental estimate of the answer. Here we see readily that the answer is something over 100; hence we locate the decimal point at the place to give us 123.0.

---

* The .5 in this answer must be estimated. Usually, if more than three significant figures are obtained from the rule, the last is uncertain.

Example 4. *Find the value of* 364 ÷ 115.

C	Set 364	Find 3.17, *Ans.*
D	over 115	over 1

Example 5. *Find the circumference of a circle whose diameter is* 42 *ft.* We multiply the diameter by $\pi = 3.14$.* Hence,

C	Set 1	Under 42
D	over 3.14	find 132.0 *Ans.*

By ordinary multiplication we get 131.88; an example of the inaccuracy of the fourth significant figure.

Example 6. *Find the continued product:* $1.6 \times 4.2 \times 5.3 \times 2.8$.

The abbreviation R. denotes the runner on the slide-rule.

C	Set 1	R. to 4.2	1 to R.	R. to 5.3	1 to R.	Under 2.8
D	over 1.6	—	—	—	—	find 99.7 *Ans.*

We add a few more rules for computing various types of expressions involving scales *A* and *B* as well as *C* and *D*.

(1) *To find* $a^2 \times b$:

A		Find a^2b, *Ans.*
B		over b.
C	Set 1	
D	over a	

(2) *To find* $a^2 \div b$:

A		Find $a^2 \div b$, *Ans.*
B	Set b	over 1.
C		
D	over a	

---

* The number π is usually marked on the scale.

(3) *To find geometric mean between two numbers a and b; i.e.* find x, so that $a/x = x/b$. Let $a < b$.

A		
B	Set a	Below b
C		
D	over a	find $x = G.\ M.$

(4) *To reduce fractions to decimals:*

C	Set numerator	Find equivalent decimal
D	over denominator	above 1

These rules are not to be memorized. They will be used almost instinctively by one who has made the reason for each rule thoroughly clear to himself and who is in practice.

EXERCISES

1. With a slide rule compute the value of:

(*a*) 2.13×4.42.

(*b*) 1.98×5.24.

(*c*) $2.77 \times 3.14 \times 4.25$.

(*d*) $8.27/2.63$.

(*e*) $5.48/3.26$.

(*f*) $10/3.14$.

(*g*) 0.000116×0.0392.

(*h*) $2{,}856{,}000 \times 256{,}700{,}000$.

(*i*) $\dfrac{5.43 \times 31.5}{21.4}$.

(*j*) $\dfrac{7.64 \times 4.14}{21.2}$.

(*k*) $\dfrac{67.4 \times 25.5 \times 19.7}{4.64 \times 18.4}$.

2. With a slide rule compute the value of:

(*a*) $(2.85)^2$.

(*b*) $3.72 \times (2.23)^2$.

(*c*) $(1.86)^3$.

(*d*) $(6.24)^2/26.3$.

3. Find the circumference and the area of a circle whose radius is 4.16 in.

4. What is the length in feet of 27.3 meters, given that 25 meters = 82 feet? Solve with one setting of the slide.

IV. LOGARITHMIC PAPER

172. Logarithmic Paper. Ruled paper is printed, on which the rulings in both directions are spaced according to the logarithmic scale (§ 169), *i.e.* precisely as on a slide rule.* Such paper is called *logarithmic paper*. Samples of this ruling are shown in Figs. 142–143.

173. Plotting Powers on Logarithmic Paper. The graphs of equations of the type

$$y = kx^n \tag{1}$$

can be plotted very readily on logarithmic paper. For, if we take the logarithms of both sides, we find

$$\log y = \log k + n \log x. \tag{2}$$

Let us set $Y = \log y$, $K = \log k$, $X = \log x$;
then (2) becomes

$$Y = K + nX. \tag{3}$$

Now the equation (3) represents *a straight line* if X and Y be taken as the variables. This is precisely what happens if we plot the values of x and y from equation (1) on logarithmic paper; for, when we plot a value for x on logarithmic paper, the *distance* from the left border is nothing else than $\log x$, *i.e.* X; and similarly for Y.

Moreover, the *slope* of the straight line represented by (3) is n, the exponent of x in (1); and the intercept on the Y axis is $K = \log k$. Hence if values of x and y from (1) are plotted on logarithmic paper, the value of n in (1) appears as the slope of the straight line graph, and the value of k can be read off directly on the vertical axis.

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* On this account, it is possible to make a crude slide rule by using the edges of two sheets of logarithmic paper, sliding them along each other after the manner of a slide rule.

Example 1. Draw the graph of the equation $y = x^2$ on logarithmic paper.

Take $x=1$, then $y = 1$. Take $x = 10$, then $y = 100$. Plot these two points A (1, 1) and B (10, 100) (Fig. 142). Connect A and B by a straight line. This is the required graph.

The graph may be drawn also by noticing that its *slope* is the exponent of x in the given equation, *i.e.* 2. Hence we may draw from A a line whose slope is 2. Show that this gives the same line, AB.

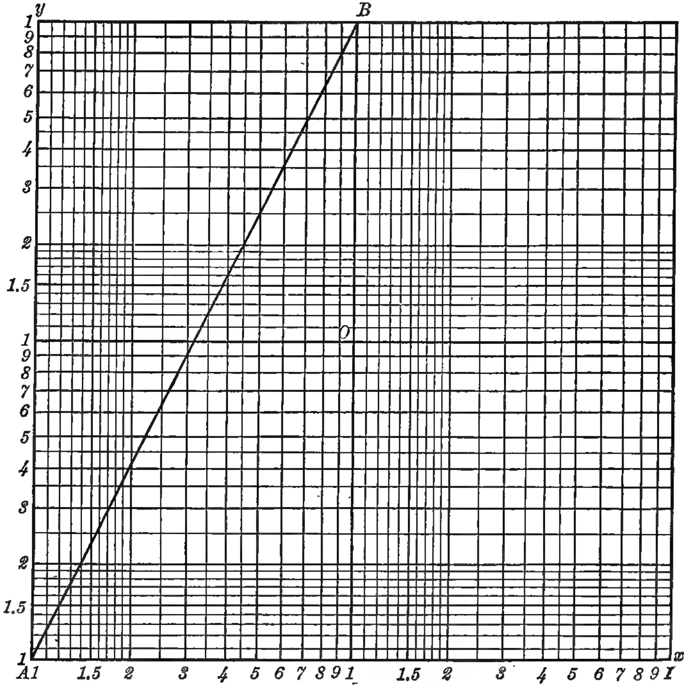

Fig. 142

We may use this graph to find squares or square roots. Thus, if $x = 4$, we can note the point on the graph directly over 4, and read the corresponding value of y, which is 16. Reversal of the process gives $\sqrt{16} = 4$. Likewise, if $x = 4.5$, we find $y = 20.2^+$; and $\sqrt{15} = 3.8$, approximately.

Conversely, given a straight line on logarithmic paper, we know that its equation must be of the form (1). We can find n by actually measuring the slope, and we can read off k on the vertical line through the point marked (1, 1), since if we place $x = 1$ in equation (2), we have $\log y = \log k$, whence $y = k$. Any other value of x may be used instead of $x = 1$, but $x = 1$ is most convenient because $\log 1 = 0$.

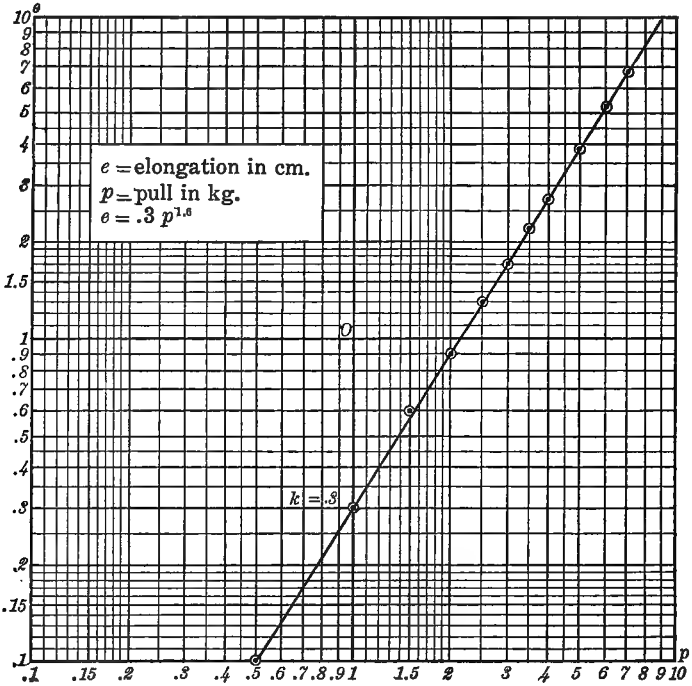

FIG. 143

EXAMPLE 2. A strong rubber band stretched under a pull of p kg. shows an elongation of e cm. The following values were found in an experiment:

p	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	6.0	7.0
e	0.1	0.3	0.6	0.9	1.3	1.7	2.2	2.7	3.3	3.9	5.3	6.9

If these values are plotted on logarithmic paper, it is evident that they lie reasonably near a straight line, such as that drawn in Fig. 143.

By measurement in the figure, the slope of this line is found to be 1.6, approximately. Hence if we set

$$P = \log p, \quad E = \log e,$$

we have

$$E = K + 1.6\,P,$$

where K is a constant not yet determined; whence

$$\log e = K + 1.6 \log p$$

or

$$e = kp^{1.6},$$

where $K = \log k$. If $p = 1$, $e = k$; from the figure, if $p = 1$, $e = 0.3$; hence $k = 0.3$, and

$$e = 0.3\,p^{1.6}.$$

EXERCISES

1. Plot on logarithmic paper the graph of each of the following equations:

(*a*) $y = x^3$. (*c*) $y = x^5$. (*e*) $y = 3\,x^2$.

(*b*) $y = x^{\frac{1}{2}}$. (*d*) $y = x^{2.5}$. (*f*) $y = 4.5\,x^{1.6}$.

2. Draw the graph of $y = x^{-2}$. Note that the negative exponent -2 gives simply what we ordinarily call a *negative* slope of -2 for the straight line graph.

3. When air expands or is compressed (as in an air compressor), without appreciable loss or gain of heat, the pressure p and the volume v are connected by the formula

$$p = kv^{-1.4}, \text{ approximately.}$$

Pressure is often measured in atmospheres, and volume in cubic feet. If we start with one cubic foot of air at one atmosphere of pressure, it is obvious that $k = 1$. Draw the graph for this case, and from it find p when $v = 0.5$ cu. ft. Find v when $p = 5$ atmospheres. Find v when $p = 0.5$ atmospheres.

4. The intercollegiate track records for foot races (1916) are as follows, where d means the distance run, and t means the record time:

d	100 yd.	220 yd.	440 yd.	880 yd.	1 mi.	2 mi.
t	$0:09\frac{4}{5}$	$0:21\frac{1}{5}$	$0:48$	$1:54\frac{4}{5}$	$4:15\frac{2}{5}$	$9:23\frac{2}{5}$

Plot the logarithms of these values on squared paper (or plot the given values themselves on logarithmic paper). Find a relation of the form $t = kd^n$. What should be the record time for a race of 1320 yd.?

(See Kennelly, *Popular Science Monthly*, Nov. 1908.)

5. In each of the following tables, the quantities are the results of actual experiments; the two variables are supposed theoretically to be connected by an equation of the form $y = kx^n$. Draw a logarithmic graph and determine k and n, approximately:

(a) (Steam pressure; v = volume, p = pressure.)

v	2	4	6	8	10
p	68.7	31.3	19.8	14.3	11.3

(Saxelby.)

(b) (Gas engine mixture; notation as above.)

v	3.54	4.13	4.73	5.35	5.94	6.55	7.14	7.73	8.05
p	141.3	115	95	81.4	71.2	63.5	54.6	50.7	45

(Gibson.)

(c) (Head of water h, and time t of discharge of a given amount.)

h	0.043	0.057	0.077	0.095	0.100
t	1260	540	275	170	138

(Gibson.)

CHAPTER X

THE IMPLICIT QUADRATIC FUNCTIONS

Two-valued Functions

I. THE FORMS $Ax^2 + Ey + C = 0$ AND $By^2 + Dx + C = 0$

174. The General Implicit Quadratic Function. We shall now return to the discussion of algebraic functions. We first discussed the explicit linear function $y = mx + b$, and the function y defined by the implicit relation $Ax + By + C = 0$ (Chapter III). Then we discussed the explicit quadratic function of the form $y = ax^2 + bx + c$ (Chapter IV). We now propose to take up the discussion of the functions y defined by *implicit* quadratic relations, such as $4y^2 - 5x = 0$, $x^2 - 4y^2 + 2x - 4y - 1 = 0$, etc. The most general form of such an equation is

$$(1) \qquad Ax^2 + Fxy + By^2 + Dx + Ey + C = 0.$$

The graphs of equations of this form are important curves, with interesting geometric properties, which we shall discuss in a later chapter. Our present purpose is to determine the general nature of these graphs (their shape, etc.) and to develop methods whereby the graph of a given equation of the type considered may be readily drawn.

We may note at the outset that the function defined by an implicit quadratic relation between x and y will usually be *two-valued, i.e.* to each value of x will correspond, in general, two distinct values of y. This is due to the fact that if any particular value be assigned to x in equation (1) above, the corresponding values of y are determined by a quadratic equation, unless $B = 0$.

We shall approach the discussion of equations of type (1) by considering in order certain simpler forms of this general type. First, we shall discuss equations of the two types

$$Ax^2 + Ey + C = 0 \text{ and } By^2 + Dx + C = 0.$$

175. The Equations $x^2 - y = 0$ and $y^2 - x = 0$. We can dispose of the equations $x^2 - y = 0$ and $y^2 - x = 0$ very quickly. The first equation is equivalent to the equation $y = x^2$, already discussed in § 72. The second equation is equivalent to the equation

$$(2) \qquad y^2 = x,$$

or

$$y = \pm\sqrt{x}.$$

We can either plot the points (x, y) whose coördinates satisfy this relation and thus obtain the graph desired*; or, we can note that the equation $y^2 = x$ is obtained from the equation $x^2 = y$ by simply interchanging x and y. Hence, the graph of $y^2 = x$ is obtained from the graph of $y = x^2$ by turning the plane of the graph of $y = x^2$ over about the line through the origin bisecting the first and third quadrants. For, this operation will interchange the x- and y-axes in the desired way. The two graphs are shown in Fig. 144.

Certain properties of the graph of the equation $y^2 = x$ are at once evident from the form of the equation: The graph is symmetric with respect to the x-axis; for, if a point (h, k) satisfies the equation, the point $(h, -k)$ also satisfies the equation. Why? The graph lies at the right of the x-axis; for, any negative value of x would give rise to imaginary values of y. Why?

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* A table of square roots will facilitate the work.

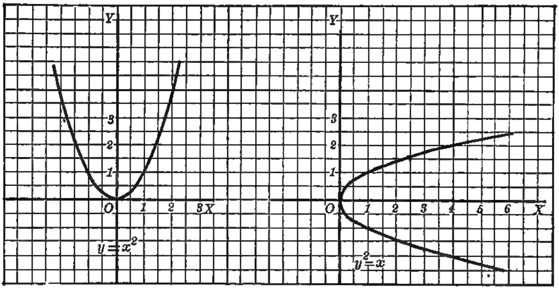

FIG. 144

The most important properties of the double-valued function $\pm\sqrt{x}$ to be noted are the following:

(1) For every positive value of x there are two values of the function, viz. $+\sqrt{x}$ and $-\sqrt{x}$. Therefore the function is two-valued.

(2) As x increases numerically, the corresponding values of $\sqrt{x}$ increase numerically, *i.e.* the numerical value of $\sqrt{x}$ is an increasing function of x.

176. The Form $By^2 + Dx = 0$. $B \neq 0$. Since $B \neq 0$, we may always write the equation in the form

$$(3) \qquad y^2 = -\frac{D}{B}x,$$

i.e. in the form

$$y^2 = nx,$$

where $n = -D/B$. The graph is then similar to that of $x^2 = ny$, the only difference being that the rôles of the x- and y-axes are interchanged. If the coefficient n is positive, the graph is at the right of the y-axis; if n is negative, the graph is at the left of the y-axis (Fig. 145). In both cases the graph is symmetric with respect to the x-axis, and

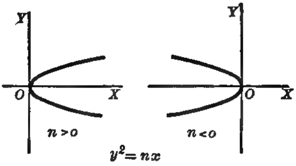

FIG. 145

passes through the origin, at which point it has a vertical tangent. Why? The curve defined by an equation of the type considered is called a ***parabola*** if $D \neq 0$. (See Chapter IV.) To sketch such a curve rapidly, knowing its general shape, we need only plot a few corresponding values of x and y. If $D=0$, the equation becomes $By^2=0$. Its graph is then the x-axis.

177. The Slope of the Curve $By^2 + Dx = 0$. To determine the slope of the tangent to the curve

$$By^2 + Dx = 0,$$

we may proceed by the method used for similar problems in Chapters IV and V. To this end we first calculate the change ratio $\Delta y/\Delta x$, which is the slope of the chord PQ (Fig. 146). The slope of the tangent at P is then the limit which this ratio approaches when Δx approaches the value 0.

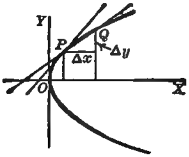

FIG. 146

Let $P(x_1, y_1)$ be any point on the curve, and $Q(x_1 + \Delta x, y_1 + \Delta y)$ be another such point. Then we have

$$B(y_1 + \Delta y)^2 + D(x_1 + \Delta x) = 0,$$

and

$$By_1{}^2 + Dx_1 = 0.$$

Expanding the first of these equations, and subtracting the second from it, we get

$$2\,By_1\Delta y + B\Delta y^2 + D\Delta x = 0,$$

or

$$(2\,By_1 + B\,\Delta y)\frac{\Delta y}{\Delta x} = -D.$$

Hence, the desired change ratio is

$$\frac{\Delta y}{\Delta x} = -\frac{D}{2\,By_1 + B\Delta y}.$$

When Δx approaches zero, Δy also approaches zero. Why? The desired slope of the curve

$$By^2 + Dx = 0$$

at the point (x_1, y_1) is, therefore,

(4) $$m = -\frac{D}{2\,By_1}.$$

The expression for the slope exhibits certain properties of the curve:

(1) The curve has a vertical tangent at the origin $(y_1 = 0)$.

(2) The slope of the curve above the x-axis is positive, if B and D have opposite signs; and negative, if B and D have the same sign.

(3) The slope of the curve decreases indefinitely in absolute value as the point (x_1, y_1) recedes indefinitely from the origin.

EXERCISES

1. For each of the following equations, determine the slope at the point (x_1, y_1) and sketch the curve represented. For each point plotted determine the slope of the tangent and draw the tangent.

(*a*) $y^2 - 4\,x = 0$; (*b*) $y^2 + 2\,x = 0$; (*c*) $4\,x^2 - 3\,y = 0$;
(*d*) $4\,y^2 + 9\,x = 0$; (*e*) $y^2 = 6\,x$.

2. Derive the equation of the tangent to each of the curves in Ex. 1 at the point indicated:

(*a*) $(1, 2)$; (*b*) $(-2, -2)$; (*c*) $(-3, 12)$; (*d*) $(-4, -3)$; (*e*) $(6, 6)$.

3. Show that the equation of the tangent to the curve $y^2 = 2\,px$ at the point (x_1, y_1) on the curve is $y_1y = p\,(x + x_1)$.

4. Draw the curves $y^2 = nx$ for several different values of n on the same sheet of paper. It is suggested that the values $n = 1, 2, 5, -1, -2, 0$ be included.

II. THE FORM $Ax^2 + By^2 + C = 0$

178. The Case $A = B$. The Equation $x^2 + y^2 = a^2$. It so happens that, if the units on the x- and y-axes are equal, we can interpret the left-hand member of this equation geometrically. For, it is evident from the figure (Fig. 147) that, under the hypothesis of equal units, $x^2 + y^2$ is the square of the distance of the point (x, y) from the origin. Hence the equation

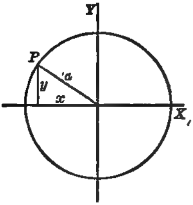

FIG. 147

$$x^2 + y^2 = a^2 \tag{5}$$

states that the point (x, y) is distant a units from the origin. It follows that the points (x, y) satisfying this equation are all on the circle described about O as center with the radius a, and conversely the coördinates of every point on this circle will satisfy the equation. The graph of the equation $x^2 + y^2 = a^2$ is then a circle, if the units on the two axes are equal.

If the units on the axes are unequal, the ordinates of the above circle must be shortened or lengthened in a certain ratio, according as the unit on the y-axis is less than or greater than the unit on the x-axis. In either case the graph of the equation will be a closed curve.

Throughout the remainder of this chapter, however, we shall assume, in order to fix ideas, that *the units on the axes are equal.*

If $A = B$ $(AB \neq 0)$, the equation

$$Ax^2 + By^2 + C = 0 \tag{6}$$

may be written in the form $x^2 + y^2 = -\frac{C}{A}$.

The graph of this equation is a circle, if $-C/A$ is positive. If $-C/A$ is negative, the equation has no graph, *i.e.* no pair of real values of x and y can satisfy it. *If* $C = 0$, the only point satisfying the equation is the origin.*

179. The Case $A > 0, B > 0$. Consider first the special case $x^2 + 4y^2 = 9$. If we solve this equation for y, we have

$$y = \pm \frac{1}{2}\sqrt{9 - x^2}. \tag{7}$$

Now, we know from § 178 that the graph of the function

$$y = \pm \sqrt{9 - x^2} \tag{8}$$

is a circle with center at the origin and radius equal to 3.

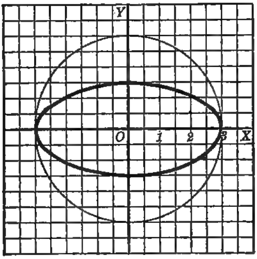

FIG. 148

The ordinates of the points of (7) are then equal to one half the corresponding ordinates of the points on the circle (8). The construction of the graph of (7) should then be clear from the figure (Fig. 148). The graph in question is a closed curve, having a greatest length of 6 units and a greatest width of 3 units. It is symmetric with respect to both axes.

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* The last locus may be considered as a circle with radius equal to 0; it is sometimes called a *point circle*.

The general form

$$(9)\qquad Ax^2 + By^2 + C = 0$$

can be treated similarly, if A and B are both positive. The equation may be written in the form

$$(10)\qquad x^2 + \frac{B}{A}y^2 = -\frac{C}{A}.$$

This shows that there is no graph if the right-hand member is negative. If the right-hand member is 0, the point (0, 0) is the only point satisfying the equation. There remains only the case where $-C/A$ is positive.

Equation (10) gives

$$(11)\qquad y = \pm\sqrt{\frac{A}{B}}\cdot\sqrt{-\frac{C}{A} - x^2}.$$

Now, the equation

$$(12)\qquad y = \pm\sqrt{-\frac{C}{A} - x^2}$$

represents a circle. Equation (11) tells us that the desired graph is obtained by shortening or lengthening the ordinates of this circle in the ratio $\sqrt{A/B}$ to 1.

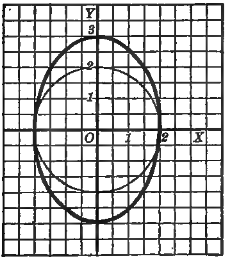

FIG. 149

EXAMPLE. If we solve the equation $9x^2 + 4y^2 = 36$ for y, we obtain $y = \pm\frac{3}{2}\sqrt{4 - x^2}$; this tells us that the graph of the given equation is obtained from that of the circle $y = \pm\sqrt{4 - x^2}$ by lengthening the ordinates of the latter to three halves their original length. Figure 149 exhibits the result.

The graph of an equation of the form $Ax^2 + By^2 + C = 0$ under the hypothesis that A and B are both positive and that C is negative, is then a closed curve symmetric with respect to both axes.

The curve represented by an equation of the form (9) above is called an ***ellipse***. An ellipse is symmetric with respect to each of two perpendicular lines, called the ***axes*** of the ellipse. The intersection of the axes of an ellipse is called the ***center*** of the ellipse. Knowing the general shape of the curve, the quickest way to sketch it from the equation is to find the intercepts on the axes and draw a symmetric curve through the four points thus obtained. In the example $9x^2 + 4y^2 = 36$ already considered, we find the intercepts to be $x = \pm 2$ (found by placing $y = 0$) and $y = \pm 3$ (when $x = 0$). If we mark the four corresponding points, the curve can be sketched readily.

EXERCISES

1. Discuss the locus of each of the following equations and, if the equation has a locus, sketch it and show how it is related to a certain circle (if the locus is not itself a circle):

(a) $x^2 + y^2 = 16$. (d) $4x^2 + y^2 + 16 = 0$. (g) $4x^2 + 3y^2 = 12$.
(b) $x^2+4y^2-16=0$. (e) $x^2 + 9y^2 = 0$. (h) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
(c) $4x^2+y^2-16=0$. (f) $2x^2 + 2y^2 = 5$.

2. For what values of x in each of the equations in Ex. 1 does y become imaginary? For what values of y does x become imaginary?

3. Show directly from the equations that each of the graphs in Ex. 1, if it exists, is symmetric with respect to both the x-axis and the y-axis.

4. According to the definition above, is a circle an ellipse?

180. The Slope of the Curve represented by $Ax^2 + By^2 + C = 0$. Here again we calculate the change ratio $\Delta y/\Delta x$, which is the slope of the secant joining the points $P(x_1, y_1)$ and $Q(x_1 + \Delta x, y_1 + \Delta y)$ on the curve, and then find the limit which this ratio approaches when Q approaches P along the curve, *i.e.* when Δx and, consequently, Δy approach 0. The calculation is as follows:

Since P and Q both lie on the curve

$$Ax^2 + By^2 + C = 0,$$

we have

(13) $$Ax_1^2 + By_1^2 + C = 0,$$

and

$$(14) \qquad A(x_1 + \Delta x)^2 + B(y_1 + \Delta y)^2 + C = 0.$$

Expanding the squares in the last equation and subtracting (13) from (14), we have

$$2\,Ax_1\Delta x + A\Delta x^2 + 2\,By_1\Delta y + B\Delta y^2 = 0,$$

or

$$(2\,By_1 + B\Delta y)\,\Delta y = -\,(2\,Ax_1 + A\Delta x)\,\Delta x,$$

whence we obtain the slope of the line PQ,

$$\frac{\Delta y}{\Delta x} = -\frac{2\,Ax_1 + A\Delta x}{2\,By_1 + B\Delta y} \qquad (B \neq 0).$$

When Δx and Δy both approach 0, we get for the slope of the curve at the point (x_1, y_1)

$$(15) \qquad m = -\frac{Ax_1}{By_1}.$$

An interesting verification of this result may be noticed. It is well known that the tangent to a circle at a point P is perpendicular to the radius OP. Now consider a circle with center at the origin. The slope of the radius through (x_1, y_1) is then clearly y_1/x_1. The slope of the tangent should, therefore, be $-x_1/y_1$. But this is exactly what the preceding formula for the slope gives, when the equation represents a circle, *i.e.* when $A = B$.

EXERCISES

1. Show from the result of the last article that at the points where the curve $Ax^2 + By^2 + C = 0$ $(ABC \neq 0)$ crosses the y-axis its tangents are horizontal; and that at the points where it crosses the x-axis its tangents are vertical.

2. Find the equation of the tangent to each of the following curves at the point indicated. Check the result by sketching the curve carefully and drawing the tangent from its equation.

(*a*) $4\,x^2 + y^2 = 25$ at $(2, 3)$. (*b*) $x^2 + 4\,y^2 = 8$ at $(2, 1)$.

(*c*) $3\,x^2 + 4\,y^2 = 16$ at $(2, -1)$.

181. The Case $A > 0$, $B < 0$. We may always write the equation (9) so that A is positive. The case where A and B have unlike signs leads to a new type of graph.

THE GRAPH OF $x^2 - y^2 = 9$. In seeking the graph of this equation, we observe first the following facts:

(1) The graph crosses the x-axis at the points $(3, 0)$ and $(-3, 0)$, and does not cross the y-axis. Why?

(2) The curve is symmetric with respect to both axes. For, if the point (h, k) is on the curve, so also is the point $(h, -k)$. Hence, the curve is symmetric with respect to the x-axis. Similarly, if the point $(h, k$ is on the curve, so also is the point $(-h, k)$. Hence the curve is symmetric with respect to the y-axis.

(3) Solving the equation for y gives us

$$y = \pm\sqrt{x^2 - 9}. \tag{16}$$

This incidentally again establishes the symmetry of the curve with respect to the x-axis. But it shows further that, if $x^2 < 9$, y is imaginary. Hence, no part of the curve lies in the strip of the plane between the lines $x = 3$ and $x = -3$. In other words all values of x between 3 and -3 are excluded. Solving the equation for x gives

$$x = \pm\sqrt{y^2 + 9}.$$

This shows that no values of y are excluded, since $y^2 + 9$ is positive for every real value of y.

(4) The slope of the curve at the point (x_1, y_1) is by § 180,

$$m = \frac{x_1}{y_1}.$$

This shows that the curve crosses the x-axis vertically, *i.e.* the lines $x = 3$ and $x = -3$ are tangent to the curve at $(3, 0)$ and $(-3, 0)$ respectively.

With these results in mind we now calculate the coördinates of a few points on the curve and the slope of the curve at these points. We thus get the following table :

x	3	4	5	6
y	0	$\sqrt{7}$	4	$3\sqrt{3}$
m	∞	$\frac{4}{7}\sqrt{7}$	$\frac{5}{4}$	$\frac{2}{3}\sqrt{3}$

We plot these points and those symmetrically situated with respect to the two axes and get Fig. 150. We know from equation (16) that y increases numerically from 0 as x increases numerically from 3. We have already seen that the curve consists of two branches. It remains only to consider what the character of the curve is for numerically large values of x.

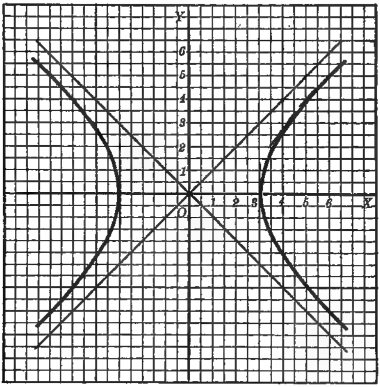

FIG. 150

Equation (16) tells us that y increases numerically without limit, as x increases indefinitely in absolute value ; *i.e.* the curve recedes indefinitely from both axes. It recedes, however, in a very definite way. For, consider the slope m of the curve at any point (x_1, y_1). From § 180 we have, for $A = 1$, $B = -1$,

$$m = \frac{x_1}{y_1} = \frac{x_1}{\pm\sqrt{x_1^2 - 9}},$$

the upper sign being used if y_1 is positive; the lower, if y_1 is negative. To fix ideas, let (x_1, y_1) be a point in the first quadrant and let it move out along the curve indefinitely. We desire to see what happens to the slope of the curve under this condition; *i.e.* when x_1 becomes indefinitely large. To this end we write m in a more convenient form, as follows:

$$m = \frac{1}{\frac{\sqrt{x_1^2 - 9}}{x_1}} = \frac{1}{\sqrt{1 - \frac{9}{x_1^2}}},$$

which shows that as x_1 increases indefinitely, m approaches more and more nearly the value $+1$. This shows that the further the point (x_1, y_1) travels out along the curve in the first quadrant, the more nearly does the direction of its motion make an angle of 45° with the x-axis.

Consider now the equation of the tangent to the curve at the point (x_1, y_1):

$$y - y_1 = \frac{x_1}{y_1}(x - x_1),$$

or,

$$x_1x - y_1y = x_1^2 - y_1^2,$$

or,

$$x_1x - y_1y = 9.$$

This may be written

$$y = \frac{x_1}{y_1} \cdot x - \frac{9}{y_1}.$$

As x_1 and y_1 become indefinitely large, the slope x_1/y_1, as we have seen, approaches the value $+1$, while the term $9/y_1$ evidently approaches the value 0. Therefore, the tangent to the curve at the point (x_1, y_1) approaches the line

$$y = x.$$

A line, which is the limiting position which the tangent to a curve approaches, as the point of contact recedes indefinitely along an infinite branch of the curve, is called an ***asymptote*** of the curve.

If the point (x_1, y_1) recedes indefinitely along the curve in the third quadrant $(x_1 < 0, y_1 < 0)$, the slope is positive and the tangent approaches the same limiting position as before, namely, $y = x$. Similar considerations (or the symmetry of the curve) show that the line

$$y = -x$$

is also an asymptote. The two asymptotes are also shown in the figure as they are a great help in drawing the curve.

THE GRAPH OF $x^2 - y^2 = a^2$. If, in place of the 9 in the equation $x^2 - y^2 = 9$ just considered, we have any other *positive* number, say a^2, the discussion is very similar and accordingly we can be brief. The curve of the equation $x^2 - y^2 = a^2$ crosses the x-axis at the points $(a, 0)$ and $(-a, 0)$, and does not cross the y-axis. It is symmetric with respect to both axes. We have $y = \pm\sqrt{x^2 - a^2}$ and $m = x_1/y_1$. We find also

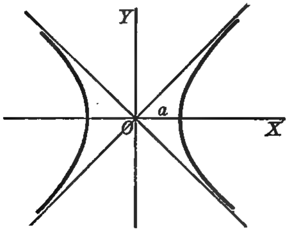

FIG. 151

$$m = \pm\frac{1}{\sqrt{1 - \frac{a^2}{x_1^2}}}$$

from which we conclude that the curve approaches indefinitely near the straight lines $y = x$ and $y = -x$. The curve is, then, as drawn in Fig. 151.

The General Case, when C is Negative. Any equation of the form

$$Ax^2 + By^2 + C = 0,$$

where A is positive and B and C are both negative, may now be treated without much difficulty. Any such equation can be written in the form

$$(17) \qquad x^2 - n^2y^2 = a^2.$$

From this we obtain

$$y = \pm \frac{1}{n}\sqrt{x^2 - a^2}.$$

But this shows at once, by comparison with the last equation considered, that the ordinates of points on the curve $x^2 - n^2y^2 = a^2$ are to the corresponding ordinates of the curve $x^2 - y^2 = a^2$ as $1/n$ is to 1. In Fig. 152 we have drawn both the curve $x^2 - y^2 = a^2$ and the curve $x^2 - 4y^2 = a^2$, the ordinates of the latter being just one half of the corresponding ordinates of the former. The asymptotes of the latter are the lines $y = \frac{1}{2}x$ and $y = -\frac{1}{2}x$.

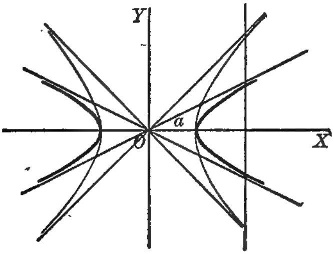

Fig. 152

Since the asymptotes are a great help in sketching the curve, we should have a means of obtaining their equations quickly from the equation of the curve. From the result of § 180 ($A = 1$, $B = -n^2$) and considerations similar to those used in the discussion of $x^2 - y^2 = 9$, we find the equations of the asymptotes to be

$$y = \frac{1}{n}x \text{ and } y = -\frac{1}{n}x,$$

or $x - ny = 0$ and $x + ny = 0$. But these equations are found by placing equal to zero *each* of the factors of the left-hand member of the equation of the curve $x^2 - n^2y^2 = a^2$.

An example will show how these various results may be applied in sketching a curve whose equation is of the form considered. To sketch the graph of $4x^2 - 9y^2 = 36$, we draw first the asymptotes $2x - 3y = 0$ and $2x + 3y = 0$ (Fig. 153). We next place $y = 0$, in the given equation and find the x-intercepts to be $x = 3$ and $x = -3$. We can now sketch the curve with considerable accuracy, since we know what its general characteristics are.

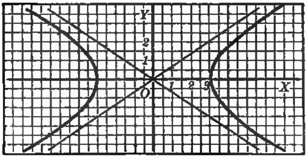

FIG. 153

The graph of any equation of the form

$$x^2 - n^2y^2 = a^2 \quad (n \neq 0,\ a \neq 0)$$

is a curve called a ***hyperbola.*** We have seen that it consists of two branches; it is symmetric with respect to each of two lines, which are called the ***axes*** of the curve. One of these cuts the curve in two points and is called the ***transverse axis***; the other axis does not meet the curve at all. The intersection of the axes of the curve is called the ***center*** of the curve. The branches of the curve extend indefinitely and approach two straight lines, the ***asymptotes*** of the curve, which pass through the center.

We may now complete the discussion of the graph of any equation of the form $Ax^2 + By^2 + C = 0$, under the hypothesis that A is positive and B negative. We have already disposed of the case $C < 0$, by considering the form $x^2 - n^2y^2 = a^2$. The case $C > 0$ leads similarly to the form $x^2 - n^2y^2 = -a^2$. By interchanging x and y this reduces to the form $n^2x^2 - y^2 = a^2$, which on division by n^2 reduces to the case $C < 0$ already considered. The graph of an equation $Ax^2 + By^2 + C = 0$, when A is positive, B negative, and C positive, is therefore a hyperbola with the center at the origin and with its transverse axis coinciding with the y-axis.

The following example will illustrate the method of sketching the curve: Sketch the graph of $x^2-4y^2+4=0$. The asymptotes are $x-2y=0$ and $x+2y=0$ (Fig. 154). Placing $x=0$, we find the y-intercepts to be $+1$ and -1. Having marked the corresponding points and drawn the asymptotes the graph is readily drawn.

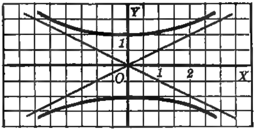

FIG. 154

Finally, when $C=0$, the equation may be written in the form $x^2-n^2y^2=0$. This may be written $(x-ny)(x+ny)=0$. This equation will be satisfied by all points which satisfy either $x-ny=0$ or $x+ny=0$, and by no others. The locus of the equation is then two straight lines passing through the origin. Figure 155 shows the locus of the equation $4x^2-9y^2=0$.

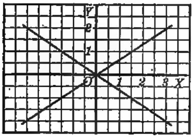

FIG. 155

182. The Case $A=0$ or $B=0$. If $A=0$, $B>0$, the equation $Ax^2+By^2+C=0$ becomes $By^2+C=0$. If C is positive, there is no graph. If C is negative, the graph consists of two lines parallel to the x-axis. If C is zero, the graph is the x-axis. When $B=0$, $A>0$, the graph of the equation consists similarly of two straight lines parallel to the y-axis, if C is negative; of the y-axis, if C is zero; and there is no graph, if C is positive.

EXERCISES

1. Sketch the graph of each of the following equations:

(a) $x^2-9y^2=16$.
(b) $x^2-9y^2=-16$.
(c) $x^2-9y^2=0$.
(d) $9x^2-16y^2+16=0$.
(e) $9x^2-16y^2-16=0$.
(f) $9x^2-16y^2=0$.
(g) $3x^2-2y^2=6$.
(h) $3x^2-12=0$.
(i) $3x^2+1=0$.

2. Give a detailed discussion of the graph of the equation $x^2-y^2=-9$ (analogous to the discussion of $x^2-y^2=9$ given in the text).

3. Give a detailed discussion of the graph of $x^2-n^2y^2=-a^2$. Prove, in particular, that the asymptotes of this hyperbola are given by $x^2-n^2y^2=0$.

4. Prove that no tangent to the curve $x^2-y^2=a^2$ has a slope that lies between $+1$ and -1. Prove, in general, that no tangent to the curve $x^2-n^2y^2=a^2$ $(a\neq 0)$ has a slope that lies between $1/n$ and $-1/n$.

III. THE FORM $Ax^2 + By^2 + Dx + Ey + C = 0$

183. Recapitulation and Extension of Previous Results. We have seen in the previous sections of this chapter that an equation of one of the forms

$$Ax^2 + Ey = 0,$$
$$By^2 + Dx = 0,$$
or
$$Ax^2 + By^2 + C = 0$$

represents either

(*a*) ***a parabola***, with vertex at the origin and axis coinciding with the x-axis or the y-axis; or,

(*b*) ***an ellipse***, with center at the origin and axes coinciding with the axes of coördinates; or

(*c*) ***a hyperbola***, with center at the origin and transverse axis coinciding with the x-axis or the y-axis; or

(*d*) ***two straight lines*** (which may coincide); or

(*e*) ***a single point*** (the point $(0, 0)$); or

(*f*) ***no locus.***

If we replace x by $x - h$ and y by $y - k$, in any of the above forms, we know that the graph of the resulting equation is obtained from the graph of the original equation by moving the latter so that the origin moves to the point (h, k) (the axes remaining parallel to their original positions).

We may then conclude that an equation of any one of the forms

$$(18) \qquad A(x - h)^2 + E(y - k) = 0, \quad B(y - k)^2 + D(x - h) = 0,$$
or
$$A(x - h)^2 + B(y - k)^2 + C = 0$$

represents either

(*a*) ***a parabola*** with vertex at the point (h, k) and axis coinciding with the line $x - h = 0$ or the line $y - k = 0$; or

(*b*) ***an ellipse*** with center at the point (h, k) and axes coinciding with the lines $x - h = 0$ and $y - k = 0$; or (*c*) ***a hyperbola*** with center at the point (h, k) and transverse axis coinciding with the line $x-h=0$, or the line $y-k=0$; or

(*d*) ***two straight lines*** (which may coincide), or

(*e*) ***a single point*** (the point (h, k)); or

(*f*) ***no locus.***

Now, any equation of the form

$$Ax^2 + By^2 + Dx + Ey + C = 0 \tag{19}$$

can be put in one of the forms (18) by completing the squares. The following examples show how this may be done.

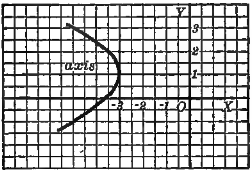

FIG. 156

EXAMPLE 1. Discuss and sketch the graph of $y^2 - 2y + 2x + 7 = 0$. This equation may be written in the form

$$y^2 - 2y = -2x - 7,$$

or

$$y^2 - 2y + 1 = -2x - 7 + 1,$$

i.e.

$$(y-1)^2 = -2(x+3).$$

It is accordingly a parabola with vertex at $(-3, 1)$ and axis $y=1$. The graph is given in Fig. 156.

EXAMPLE 2. Discuss and sketch the graph of $x^2 + y^2 - 4x - 6y + 9 = 0$.

This equation may be written in the form

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -9 + 4 + 9,$$

or

$$(x-2)^2 + (y-3)^2 = 4.$$

Therefore the given equation represents a circle with center at $(2, 3)$ and radius equal to 2. (See Fig. 157.)

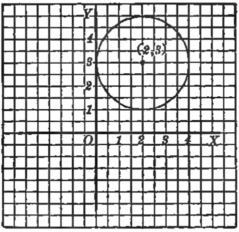

FIG. 157

Example 3. Discuss and sketch the graph of $9x^2 + 16y^2 - 18x + 64y - 8 = 0$.

This equation may be written in the form

$$9(x^2 - 2x + \quad) + 16(y^2 + 4y + \quad) = 8,$$

or

$$9(x^2-2x+1)+16(y^2+4y+4)=8+9+64=81,$$

i.e.

$$9(x-1)^2 + 16(y+2)^2 = 81.$$

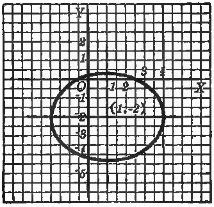

Fig. 158

Hence this equation represents an ellipse whose center is at (1, − 2) and whose axes coincide with the lines $x = 1$, $y = -2$. The remainder of the discussion is left as an exercise. The graph is given in Fig. 158.

Example 4. Discuss and sketch the graph of $9x^2 - 36x - 4y^2 + 24y = 36$.

This equation may be written in the form

$$9(x-2)^2 - 4(y-3)^2 = 36,$$

which is a hyperbola whose center is at (2, 3) (Fig. 159). It is left as an exercise to complete the discussion and prove that the equations of the asymptotes are $3(x-2) + 2(y-3) = 0$ and $3(x-2) - 2(y-3) = 0$.

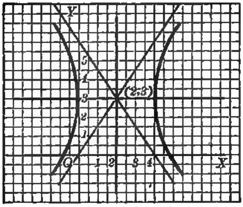

Fig. 159

EXERCISES

Discuss and sketch the graph of each of the following equations:

1. $x^2 + 4y + 4 = 0$.
2. $x^2 + y^2 + 4x - 8y + 1 = 0$.
3. $x^2 - y^2 + 2x = 0$.
4. $x^2 - 4x + y^2 + 2y + 1 = 0$.
5. $x^2 + 4x + 2y^2 + 4y + 1 = 0$.
6. $9x^2 + 4y^2 - 36x - 8y + 4 = 0$.
7. $9x^2 - 4y^2 - 36x + 8y = 4$.
8. $y^2 + 2y - 12x - 11 = 0$.
9. $x^2 + 15y^2 + 4x + 60y + 15 = 0$.
10. $x^2 - 3y^2 - 2x - 6y + 7 = 0$.

184. The Slope of the Curve $Ax^2 + By^2 + Dx + Ey + C = 0$. Let $P(x_1, y_1)$ and $Q(x_1 + \Delta x, y_1 + \Delta y)$ be any two points on the curve. Then

$$Ax_1^2 + By_1^2 + Dx_1 + Ey_1 + C = 0,$$

$$A(x_1 + \Delta x)^2 + B(y_1 + \Delta y)^2 + D(x_1 + \Delta x) + E(y_1 + \Delta y) + C = 0.$$

Expanding the second of these equations and subtracting the first from it, we have

$$(2\,Ax_1 + A\Delta x + D)\Delta x + (2\,By_1 + B\Delta y + E)\Delta y = 0.$$

Therefore the change ratio, or the slope, of the secant PQ, is

$$\frac{\Delta y}{\Delta x} = -\frac{2\,Ax_1 + A\Delta x + D}{2\,By_1 + B\Delta y + E}.$$

If we let Δx approach zero, Δy will approach zero also. Why? Therefore the slope of the curve at any point (x_1, y_1) is

$$m = -\frac{2\,Ax_1 + D}{2\,By_1 + E}.$$

Example. Find the equations of the tangent and the normal to the curve $x^2 + 4\,y^2 - 4\,x + 2\,y - 3 = 0$ at the point (1, 1).

Solution: The slope of the tangent at any point (x_1, y_1) is

$$m = -\frac{2\,x_1 - 4}{8\,y_1 + 2}.$$

At the point (1, 1) this slope is $\frac{1}{5}$. Therefore the equation of the tangent is $y - 1 = \frac{1}{5}(x - 1)$ and the equation of the normal is $y - 1 = -5(x - 1)$.

EXERCISES

1. Find the slope of the tangent to each of the following curves at the point specified.

(a) $x^2 + 2\,y - 3 = 0$ at (1, 1);
(b) $x^2 + y^2 - 4 = 0$ at $(1, \sqrt{3})$;
(c) $x^2 - 2\,y^2 + 5 = 0$ at $(1, \sqrt{3})$;
(d) $4\,x^2 + y^2 - 2\,x - 3\,y - 10 = 0$ at (2, 1).

2. Find the equation of the tangent to each of the curves of Ex. 1, at the point specified.

IV. THE FORM $Fxy + Dx + Ey + C = 0$

185. The Graph of $xy = a$. The graph of the curve

$$xy = a$$

is symmetric with respect to the origin; for, if the coördinates (h, k) satisfy the equation, the coördinates $(-h, -k)$ also satisfy it. Since $y = a/x$, it is evident that x may assume all values except 0. (See § 36.) As x increases numerically without limit, the curve approaches the line $y = 0$, *i.e.* $y = 0$ is an asymptote. Similarly as y increases without limit, the curve approaches the line $x = 0$ as an asymptote. It will be proved later that the curve is a hyperbola, provided a is not equal to zero. If a is positive, the graph is as in Fig. 160. If a is negative, the graph is as in Fig. 161. If a is zero, the graph consists of the two axes $x = 0$ and $y = 0$.

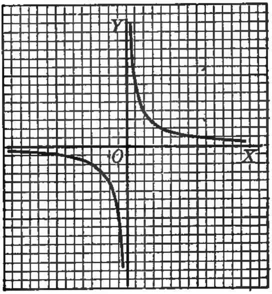

FIG. 160

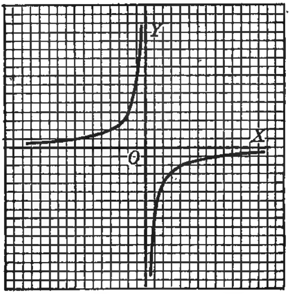

FIG. 161

186. The Graph of $Fxy + Dx + Ey + C = 0$. If in the equation $xy = a$ we replace x by $x - h$ and y by $y - k$, we know that the graph of the resulting equation is obtained from the graph of the original equation by moving the latter so that the origin moves to the point (h, k), the axes remaining parallel to their original positions. It follows that the equation

$$(x - h)(y - k) = a \ (a \neq 0)$$

represents a hyperbola whose asymptotes are $x = h$, $y = k$. If $a = 0$, the equation represents the two lines $x = h$, $y = k$.

Example. Discuss and sketch the graph of $xy + 4x + 2y = 1$.

First we write

$$(x \pm ?)(y \pm ?) = 1.$$

Then from inspection we see that the given equation may be written in the form

$$(x + 2)(y + 4) = 9.$$

That is, the graph is a hyperbola whose asymptotes are $x = -2$, $y = -4$. (See Fig. 162.)

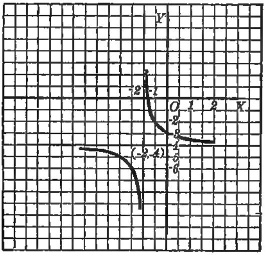

Fig. 162

187. The Slope of the Curve $Fxy + Dx + Ey + C = 0$. It is left as an exercise to show that the slope of the curve

$$Fxy + Dx + Ey + C = 0$$

at any point (x_1, y_1) is

$$m = -\frac{Fy_1 + D}{Fx_1 + E}.$$

EXERCISES

1. Discuss and draw the graph of each of the following curves:

(a) $xy = 1$; (b) $xy = -1$; (c) $xy = 2$; (d) $xy = -2$.

2. Discuss and draw the graph of each of the following curves.

(a) $xy + 2x = 8$; (b) $xy + 2x + 4y = 8$; (c) $xy - 4x + 8y = 2$.

3. Draw the family of curves $xy = a$, taking several positive and several negative values of a. How does $xy = 0$, compare with these?

4. Show that any equation of the form

$$y = \frac{ax + b}{cx + d}$$

can be reduced to the form given in § 186.

V. THE GENERAL FORM $Ax^2 + Fxy + By^2 + Dx + Ey + C = 0$

188. The Graph. Methods of drawing the graph of an equation in the above form will be illustrated by means of the following examples.

Example 1. Discuss and sketch the graph of

$$x^2 + 2xy + y^2 - 2x - 2 = 0.$$

Solving for y, we have $y = -x \pm \sqrt{2x+2}$. All values of x less than -1 must be excluded, for these values make $2x+2$ negative. Similarly, since $x = -(y-1) \pm \sqrt{-2y+3}$, it follows that all values of y greater than $\frac{3}{2}$ must be excluded; for these values make $-2y+3$ negative. The x-intercepts are the roots of the equation $x^2 - 2x - 2 = 0$, *i.e.* $1 \pm \sqrt{3}$. The y-intercepts are the roots of the equation $y^2 - 2 = 0$, *i.e.* $\pm\sqrt{2}$. From $y = -x \pm \sqrt{2x+2}$ it is seen that x may start with the value -1 and increase without limit. Similarly from $x = -(y-1) \pm \sqrt{-2y+3}$ we see that y may start with the value $\frac{3}{2}$ and decrease without limit. Using the above data and plotting the points

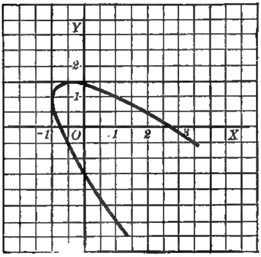

Fig. 163

x	-1	0	1	2	$1 \pm \sqrt{3}$
y	1	$\pm\sqrt{2}$	$1, -3$	$-2 \pm \sqrt{6}$	0

we obtain the graph in Fig. 163.

This problem may be approached from an entirely different standpoint. Suppose we let $y' = \pm\sqrt{2x+2}$ and $y'' = -x$. Plotting these curves* (Fig. 164), adding the ordinates of $y' = \pm\sqrt{2x+2}$ to the ordinates of $y'' = -x$, gives us the desired graph. This may be done graphically. We have here a shear of $y' = \pm\sqrt{2x+2}$ with respect to the line $y'' = -x$. (See § 90.)

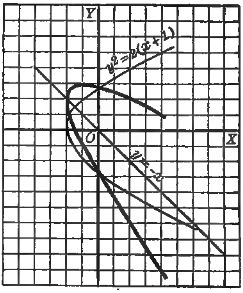

Fig. 164

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* Observe that the equation $y' = \pm\sqrt{2x+2}$ is equivalent to $y'^2 = 2(x+1)$.

Example 2. Discuss and sketch the graph of

$$y^2 - 2xy + 2x^2 - 5x + 4 = 0.$$

Solving for y, we have

$$y = x \pm \sqrt{-x^2 + 5x - 4}.$$

Hence, we merely have to shear the circle

$$y' = \pm\sqrt{(x-1)(4-x)},$$

i.e.

$$x^2 + y^2 - 5x + 4 = 0,$$

with respect to the line $y'' = x$ in order to obtain the desired result. (See Fig. 165.) The complete discussion is left as an exercise.

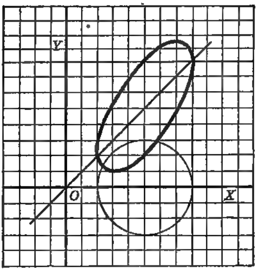


Fig. 165

Example 3. Discuss and sketch the graph of

$$7x^2 + 36xy - 36y^2 - 25 = 0.$$

Solving for y, we have

$$y = \tfrac{1}{2}x \pm \tfrac{1}{6}\sqrt{16x^2 - 25},$$

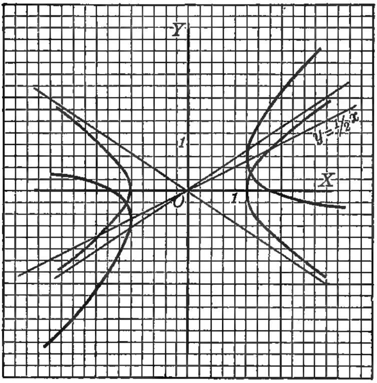


Fig. 166

which shows that the desired graph may be obtained by shearing

$$y = \pm\tfrac{1}{6}\sqrt{16x^2 - 25},$$

i.e.

$$16x^2 - 36y^2 - 25 = 0,$$

with respect to the line $y = \frac{1}{2}x$ (See Fig. 166.) The complete discussion is left as an exercise.

U

EXERCISES

Discuss and sketch the graph of each of the following equations :

1. $4x^2 + y^2 - 4xy - x + 3 = 0.$

2. $y^2 - 2xy + 3x = 2.$

3. $y^2 - 8xy + 16x^2 = 1 - x^2.$

4. $4y^2 - 4xy + x^2 = 1 - x.$

5. $9y^2 - 12xy + 3x^2 + 5x = 6.$

6. $y^2 - 6xy + 8x^2 - 10x - 25 = 0.$

189. The Slope of the Curve $Ax^2 + By^2 + Fxy + Dx + Ey + C = 0$. It is left as an exercise to prove that the slope m at any point (x_1, y_1) is

$$m = -\frac{2Ax_1 + Fy_1 + D}{2By_1 + Fx_1 + E}.$$

EXERCISES

Find the equations of the tangent and the normal to each of the following curves at the points indicated.

1. $48x^2 - 11xy - 17y^2 - 129x + 24y + 81 = 0$; $(2, 1)$, $(3, -3)$.

2. $xy + 2x - x^2 + y^2 + 6y = 0$; $(0, 0)$, $(0, -6)$.

3. $81y^2 + 72xy + 16x^2 - 96x = 378y - 144$; $(3, 2)$.

190. A General Theorem. The results of the examples and exercises of § 188 suggest that the graphs of equations of the second degree involving an xy-term are similar to the graphs of equations of the second degree in which the xy-term is lacking. We may now prove that this is a fact. The theorem is as follows :

Any equation of the form $Ax^2 + Fxy + By^2 + Dx + Ey + C = 0$ *represents either an ellipse, or a hyperbola, or a parabola, or two straight lines (which may coincide), or a single point, or no locus.*

We shall prove this theorem by showing that if the locus of the equation

$$Ax^2 + Fxy + By^2 + Dx + Ey + C = 0 \tag{20}$$

be rotated about the origin through a *properly chosen angle* θ, its equation will be of the form

$$A'x^2 + B'y^2 + D'x + E'y + C = 0. \tag{21}$$

The theorem then follows from § 183.

We saw in § 137 that, if any point $P(x, y)$ be rotated about the origin through an angle θ to a new position $P'(x', y')$, the coördinates of P and P' are connected by the relations:

$$\begin{aligned} x' &= x \cos\theta - y \sin\theta, \\ y' &= x \sin\theta + y \cos\theta. \end{aligned} \tag{22}$$

Solving these equations for x and y in terms of x', y', we obtain

$$\begin{aligned} x &= x' \cos\theta + y' \sin\theta, \\ y &= - x' \sin\theta + y' \cos\theta. \end{aligned} \tag{23}$$

If $P(x, y)$ satisfies equation (20), $P'(x', y')$ will satisfy the equation obtained by substituting the values of x, y from (23) in equation (20).

The result of this substitution is as follows:

$$\begin{aligned} A\,(x' \cos\theta + y' \sin\theta)^2 + F(x' \cos\theta + y' \sin\theta)&(-x' \sin\theta + y' \cos\theta) \\ &+ B(-x' \sin\theta + y' \cos\theta)^2 \\ &+ D(x' \cos\theta + y' \sin\theta) \\ &+ E(-x' \sin\theta + y' \cos\theta) + C = 0. \end{aligned}$$

When expanded and rearranged according to the terms in x', y', we obtain

$$A'x'^2 + F'x'y' + B'y'^2 + D'x' + E'y' + C = 0, \tag{24}$$

where
$$\begin{aligned} A' &= A \cos^2\theta + B \sin^2\theta - F \sin\theta \cos\theta. \\ F' &= 2(A - B) \sin\theta \cos\theta + F(\cos^2\theta - \sin^2\theta). \\ B' &= A \sin^2\theta + B \cos^2\theta + F \sin\theta \cos\theta. \\ D' &= D \cos\theta - E \sin\theta. \\ E' &= D \sin\theta + E \cos\theta. \\ C' &= C. \end{aligned}$$

Equation (24) will be of the desired form (21), if the angle θ is so chosen that $F' = 0$. Now, F' may be written

$$F' = (A - B) \sin 2\theta + F \cos 2\theta: \tag{25}$$

F will, therefore, be equal to zero, if

$$\tan 2\theta = \frac{F}{B - A}.$$

A value of θ satisfying the condition (25) can then always be found.* This completes the proof of the theorem.

The following exercises will illustrate the above proof. The method may also be used to draw the graphs of equations involving the xy-term.

---

* If $B = A$, we take $2\theta = 90°$, *i.e.* $\theta = 45°$.

EXERCISES

Determine the angle θ through which the loci of the following equations must be rotated in order that their new equations shall contain no xy-term. Determine the new equation and use it to draw the locus of the original.

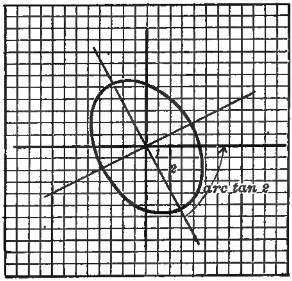
1. $8x^2 + 4xy + 5y^2 - 36 = 0$.

Solution: After substituting $x = x' \cos\theta + y' \sin\theta$, $y = -x' \sin\theta + y' \cos\theta$, the equation becomes

$$
\begin{aligned}
(1) \qquad & (8\cos^2\theta + 5\sin^2\theta - 4\sin\theta\cos\theta)x'^2 \\
& + [6\sin\theta\cos\theta + 4(\cos^2\theta - \sin^2\theta)]x'y' \\
& + (8\sin^2\theta + 5\cos^2\theta + 4\sin\theta\cos\theta)y'^2 - 36 = 0.
\end{aligned}
$$

Therefore,
$$\tan 2\theta = -\frac{4}{3} = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Solving this equation for $\tan\theta$, we have
$$4\tan^2\theta - 6\tan\theta - 4 = 0,$$
or
$$\tan\theta = 2 \text{ or } -\tfrac{1}{2}.$$



We choose $\tan\theta = 2$ (θ in first quadrant); therefore
$$\sin\theta = \frac{2}{\sqrt{5}}, \quad \cos\theta = \frac{1}{\sqrt{5}}.$$

Substituting these values in (1) we obtain
$$4x'^2 + 9y'^2 = 36.$$

The desired graph is obtained from the graph of this equation by rotating it through the angle $-\theta$ about the origin. The construction of the adjacent figure explains itself.

2. $x^2 - y^2 + 2xy - 12 = 0$.

3. $x^2 - y^2 + 2xy + 2x - 12 = 0$.

4. $xy = 4$.

5. $3x^2 - 2xy + y^2 + 6 = 0$.

6. $8x^2 - 12xy + 3y^2 - 36 = 0$.

7. $2x^2 - 12xy - 3y^2 + 42 = 0$.

8. $5x^2 + 4xy - y^2 + 48x - 12y - 10 = 0$.

9. $9y^2 + x^2 + 2xy = 0$.

10. Prove that the locus of $xy = c$ may be rotated about the origin so as to coincide with the locus of $x^2 - y^2 = a^2$, provided $a^2 = \pm 2c$.

11. With the notation of § 190, prove that $A' + B' = A + B$ and that $(A' - B')^2 + F'^2 = (A - B)^2 + F^2$.

PART III. APPLICATIONS TO GEOMETRY

CHAPTER XI

THE STRAIGHT LINE

191. Introduction. We have hitherto used coördinates primarily for the purpose of representing functions graphically and investigating the properties of those functions. We have seen that every continuous function defines a curve or a straight line, the graph of the function. Thus far, we have laid emphasis only on the discovery of the characteristics of the functions from the known properties of the curves that represent them.

Conversely, we have seen that every curve or straight line, in the plane of a system of rectangular coördinates, defines a function; *i.e.* the points of any such curve associate with every value of x one or more values of y. If this function can be determined when the curve is given, the properties of the curve may be studied from the properties of the function. This function is usually expressed by means of an equation in x and y, called the *equation of the curve.* We propose now to study the properties of various curves by means of their equations. (See § 62.)

Up to this time, we have used different scales on the two axes whenever it was convenient to do so. *Throughout this and the next four chapters we shall assume, unless the contrary is specifically stated, that* ***the units on the x- and y-axes are equal.***

192. The Distance between two Points. Given the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, let us find the length of the segment $\overline{P_1P_2}$. If a line be drawn through P_1 parallel to the x-axis and another through P_2 parallel to the y-axis to form the right triangle P_1QP_2 (Fig. 167), we have at once

$$(1) \qquad \overline{P_1P_2} = \sqrt{\overline{P_1Q}^2 + \overline{QP_2}^2}.$$

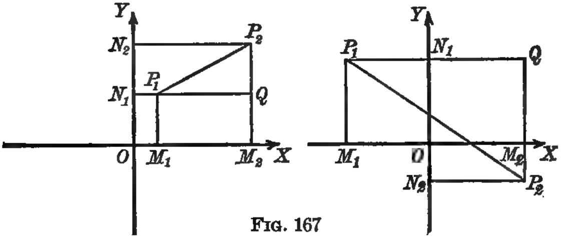


FIG. 167

The segment $\overline{P_1Q}$ is equal to the projection $\overline{M_1M_2}$ of $\overline{P_1P_2}$ on the x-axis and $\overline{QP_2}$ is equal to the projection $\overline{N_1N_2}$ of $\overline{P_1P_2}$ on the y-axis. By the result of § 37, we have

$$\overline{P_1Q} = \overline{M_1M_2} = x_2 - x_1,$$
$$\overline{QP_2} = \overline{N_1N_2} = y_2 - y_1.$$

Substituting these values in (1), we have the desired formula:

$$(2) \qquad \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

193. The Simple Ratio. Given two distinct points P_1, P_2 and any point P (distinct from P_2) on the line P_1P_2, the ratio $\overline{P_1P}/\overline{PP_2}$ is called the *simple ratio* of P with respect to P_1, P_2.

The line-segments in this definition are *directed* segments. Accordingly the simple ratio of P with respect to P_1, P_2 is positive if P is between P_1 and P_2, and negative if P is on either prolongation of the segment P_1P_2.

194. Point of Division. The coördinates (x, y) of the point P on the line joining $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ such that the simple ratio

$$\frac{\overline{P_1P}}{\overline{PP_2}} = \lambda,$$

are given by the formulas

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}. \tag{3}$$

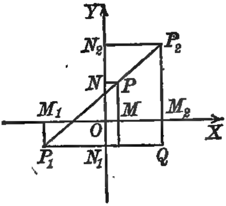


FIG. 168

PROOF. Draw lines through P_1, P_2, P parallel to the axes, meeting the x-axis in M_1, M_2, M, and the y-axis in N_1, N_2, N, respectively (Fig. 168). Then, since $\overline{P_1P}/\overline{PP_2} = \lambda$, we have

$$\frac{M_1M}{MM_2} = \lambda, \quad \frac{N_1N}{NN_2} = \lambda.$$

The first of these relations gives (by § 37)

$$\frac{x - x_1}{x_2 - x} = \lambda.$$

Solving this equation for x gives

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}.$$

Similarly from the second relation above we obtain

$$y = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

The mid-point of P_1P_2 is obtained from the value $\lambda = 1$. Why? Accordingly the coördinates of the midpoint of P_1P_2 are

$$\left(\frac{x_1 + x_2}{2}, \quad \frac{y_1 + y_2}{2}\right).$$

EXERCISES

1. Find the distance between the following pairs of points: $(1, 2)$ and $(5, 3)$; $(-1, 6)$ and $(2, -3)$; $(-2, -1)$ and $(-1, 4)$; $(-3, 4)$ and $(1, 4)$.

2. Find the lengths of the sides of the triangle whose vertices are $(-1, 1)$, $(4, -4)$, and $(1, 3)$. Prove that it is a right triangle.

[HINT: A right triangle is the only kind of triangle in which the square of one side is equal to the sum of the squares of the other two sides.]

3. An isosceles triangle has its vertex at $(4, 4)$ and the vertex of one of its base angles at $(0, -1)$. The vertex of the other base angle is on the x-axis. Find the coördinates of the latter vertex.

[HINT: Let the unknown point be $(x, 0)$ and equate the equal sides. How many solutions are there?]

4. Find the coördinates of the point whose simple ratio with respect to $(2, 1)$ and $(-4, 7)$ is 2. Find the coördinates of another point whose simple ratio with respect to the same two given points is -2.

Draw a figure illustrating this problem.

5. Check the result of Ex. 4 by calculating the lengths of the segments involved.

6. Find the coördinates of the point which divides the segment from $(2, -1)$ to $(-4, 3)$ internally in the ratio $1:4$.

7. Find the coördinates of the mid-points of the sides of the triangle in Ex. 2.

8. A quadrilateral has its vertices at the points $(-2, 1)$, $(3, 1)$, $(5, 3)$, and $(0, 3)$. Show that its diagonals bisect each other. What kind of a quadrilateral is it?

9. Find the coördinates of the points of trisection of the segment from $(3, -6)$ to $(0, 3)$.

10. A triangle has its vertices at the the points $(0, 4)$, $(2, -6)$, $(-2, -2)$. Find the coördinates of the points two thirds of the way from each vertex to the middle point of the opposite side, and thus show that the three medians of the triangle all pass through the same point.

11. The vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3). Find the coördinates of the point of intersection of the medians.

12. Show that the triangle $A(4, 1)$, $B(1, 4)$, $C(5, 5)$ is isosceles.

13. One end of a line whose length is 13 units is at the point (3, 8). The ordinate of the other end is 8. What is its abcsissa?

14. The middle point of a line is (2, 3) and one end of the line is at the point (4, 7). What are the coördinates of the other end?

15. The points (2, 1), (3, 4), $(-1, 7)$ are the mid-points of the sides of a triangle. Find the coördinates of the vertices.

16. Find the area of the isosceles triangle whose vertices are (4, 1), (1, 4), (5, 5) by finding the length of the base and the altitude.

17. What equation must be satisfied if the points (x, y), (2, 1), (1, 4) form an isosceles triangle the equal sides of which meet in (x, y)?

18. Prove that the points $(-2, -1)$, (1, 0), (4, 3) and (1, 2) are the vertices of a parallelogram.

19. The line from (x_1, y_1) to (x_2, y_2) is divided into 5 equal parts. Find the coördinates of the points of division.

20. A point is equidistant from the points (2, 1) and $(-2, 1)$ and 7 units distant from the origin. Find its coördinates.

QUESTIONS FOR DISCUSSION

1. Does the distance between two points depend on the order in which the points are taken? Does the formula for the distance give the same result no matter in which order the points are taken? Why?

2. Does the simple ratio of a point with respect to P_1, P_2 depend on the order in which the points P_1, P_2 are taken? What is the relation between the simple ratio of P with respect to P_1, P_2 and the simple ratio of P with respect to P_2, P_1?

[HINT. The answer to this question follows most easily from the definition of simple ratio. Prove the relation in question by means of the formulas in § 194.]

3. Can the simple ratio of a point P with respect to P_1, P_2 be -1? Why? As the simple ratio approaches -1 what is the motion of P?

4. What can be said of the position of the point P, if its simple ratio with respect to P_1, P_2 is positive? if its simple ratio lies between 0 and -1? if its simple ratio is less than -1?

5. If the simple ratio of P with respect to P_1, P_2 is λ, what is the simple ratio of P_1 with respect to P and P_2? of P_2 with respect to P_1 and P?

195. The Area of a Triangle. One Vertex at the Origin. Let us try to find the area of a triangle whose vertices are $O(0, 0)$, $P_1(x_1, y_1)$, and $P_2(x_2, y_2)$. Let the angles XOP_1 and XOP_2 be denoted by θ_1 and θ_2, respectively, and let the angle P_1OP_2 of the triangle have the absolute measure θ (Fig. 169).

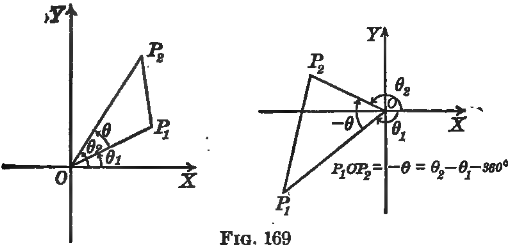


FIG. 169

The area of the triangle is then equal to $\frac{1}{2}\, OP_1 \cdot OP_2 \sin\theta$. Now, the directed angle P_1OP_2 differs from $\theta_2 - \theta_1$, if at all, only by multiples of 360° (§ 101). Therefore

$$\sin\theta = \pm \sin(P_1OP_2) = \pm \sin(\theta_2 - \theta_1).$$

The area of the triangle OP_1P_2 is, then,

$$\begin{aligned} A &= \pm \tfrac{1}{2}\, OP_1 \cdot OP_2 \sin(\theta_2 - \theta_1) \\ &= \pm \tfrac{1}{2}\, OP_1 \cdot OP_2 (\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1) \qquad (\S\ 138) \\ &= \pm \tfrac{1}{2}\, OP_1 \cdot OP_2 \left(\frac{y_2}{OP_2} \cdot \frac{x_1}{OP_1} - \frac{x_2}{OP_2} \cdot \frac{y_1}{OP_1}\right) \\ &= \pm \tfrac{1}{2}(y_2x_1 - x_2y_1). \end{aligned}$$

The area of the triangle OP_1P_2, in the ordinary sense of the term, is therefore equal to the *absolute value* of the expression

$$\tfrac{1}{2}(x_1y_2 - x_2y_1).$$

For some purposes it is convenient, however, to regard the area enclosed by a curve as a signed quantity, just as we have found it convenient to regard line-segments and angles as signed quantities.

To this end, we observe that a point moving on the boundary of an area may make the circuit in either of two opposite directions (Fig. 170). With each of these directions is associated a definite rotation about a point within the area. If the boundary is traversed in a direction which produces a positive rotation about a point within the area, the circuit and the area are regarded as positive; if the boundary is traversed in the opposite direction, the circuit and the area are regarded as negative. Hence if an area is represented by a signed number, the sign of this number tells us the direction in which the boundary is traversed.

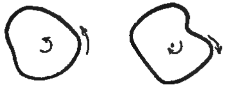


FIG. 170

In case of a triangle OP_1P_2 (Fig. 169) the order in which the vertices are written determines a direction of traversing the boundary. If OP_1P_2 is positive, OP_2P_1 is negative, and *vice versa*. Now in going around the triangle in the direction OP_1P_2, a segment OP joining O to a point P moving on P_1P_2 generates a directed angle P_1OP_2. This angle is positive or negative according as the circuit OP_1P_2 is positive or negative. Moreover the measure of the angle P_1OP_2 differs from $\theta_2 - \theta_1$, if at all, only by multiples of 360°. The expression

$$\tfrac{1}{2}\, OP_1 \cdot OP_2 \sin(\theta_2 - \theta_1)$$

is, therefore, positive or negative according as the circuit OP_1P_2 is positive or negative. We have then finally:

The area of a triangle OP_1P_2 is given in magnitude and in sign by the formula

$$\textbf{Area } OP_1P_2 = \tfrac{1}{2}(x_1y_2 - x_2y_1). \tag{4}$$

196. The Area of Any Triangle. The convention as to the sign of an area is serviceable in deriving a formula for the area of any triangle in terms of the coördinates of its vertices. Let the vertices be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$. Join these vertices to the origin by lines OP_1, OP_2, OP_3. We now consider the three possible cases, according as the origin is inside (Fig. 171), outside (Fig. 172), or on, a side (Fig. 173) of the triangle $P_1P_2P_3$. Then *in all cases*, we have

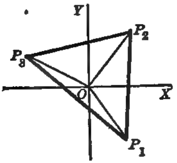


FIG. 171

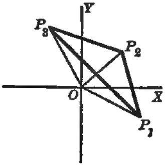


FIG. 172

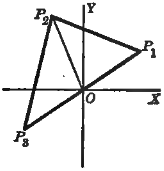


FIG. 173

$$\triangle\, P_1P_2P_3 = \triangle OP_1P_2 + \triangle OP_2P_3 + \triangle OP_3P_1,$$

if due regard is taken of the signs of the areas. Hence

$$\textbf{(5) Area of } \triangle\, P_1P_2P_3 = \tfrac{1}{2}(y_2x_1 - x_2y_1 + y_3x_2 - x_3y_2 + y_1x_3 - x_1y_3).$$

It might appear that this formula is difficult to apply. The following method makes it very simple. Write the coördinates of the vertices in two vertical columns as indicated, repeating the coördinates of the first vertex. Multiply each x by the y in the next row and add the products. This gives $x_1y_2 + x_2y_3 + x_3y_1$. Then multiply each y by the x in the next row and add the products. This gives $y_1x_2 + y_2x_3 + y_3x_1$. Subtract the second sum from the first and *divide the result by* 2. The final result will be the area sought, with its proper sign.* A similar method may be used for finding the area of any convex polygon whose vertices are given. See Exs. 6, 7, 8, pp. 301, 302.

$$\begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{array}$$

---

* The student familiar with the elements of the theory of determinants will observe that the area can be expressed as

$$\Delta = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Example. Find the area of the triangle whose vertices are $P_1(-4, 3)$, $P_2(-1, -2)$, $P_3(-3, -1)$. We write the coördinates of the vertices in two columns, repeating those of the first vertex. Performing the first step described in the previous paragraph we obtain $8 + 1 - 9 = 0$; the second step gives $-3 + 6 + 4 = 7$; the third step gives $0 - 7 = -7$; dividing this by 2, we obtain $-3\frac{1}{2}$ as the area of triangle $P_1P_2P_3$. The magnitude of the area is $3\frac{1}{2}$ square units, and the direction P_1 to P_2 to P_3 is negative. Draw the figure and verify the latter statement.

$$\begin{array}{rr} -4 & 3 \\ -1 & -2 \\ -3 & -1 \\ -4 & 3 \end{array}$$

197. Condition for Collinearity of three Points.

If three points P_1, P_2, P_3 are collinear, the area of the triangle formed by them is zero; conversely, if the area of a triangle is zero, the three vertices are collinear. Therefore, *a necessary and sufficient condition that three points be collinear, is that the right hand member of* (5), p. 300 *be equal to zero.*

EXERCISES

1. Find the areas of the following triangles and interpret the sign of the result in each case. Illustrate by appropriate figures.

(*a*) $(1, 3), (4, 2), (2, 5)$. (*c*) $(-5, 2), (-4, -3), (1, -1)$.
(*b*) $(2, 4), (-3, 1), (1, -7)$. (*d*) $(a, a), (-b, -b,)\ (c, d)$.

2. Show that the following sets of three points are collinear:

(*a*) $(0, 1), (2, 5), (-1, -1)$. (*c*) $(1, -2), (6, 1), (-4, -5)$.
(*b*) $(2, 1), (-4, 4), (4, 0)$. (*d*) $(0, -b), (1, a-b), (a, a^2-b)$.

3. The point (h, h) is collinear with $(2, 5)$ and $(5, -3)$. Find its coördinates.

4. Find the point on the y-axis collinear with $(2, 5)$ and $(5, -3)$.

5. Under what conditions on a, b, c, and d are the points in Ex. 1 (*d*) collinear? Interpret each of the conditions geometrically.

6. *Area of any polygon.* Show that the method of § 196 may be extended to derive a formula for the area of any polygon in which two sides do not cross each other, and that if $P_1P_2P_3 \cdots P_n$ are the vertices of the polygon taken in order around the polygon, we have

$$\text{Area of polygon} = \triangle\, OP_1P_2 + \triangle\, OP_2P_3 + \triangle\, OP_3P_4 + \cdots + \triangle\, OP_nP_1,$$

if due regard is paid to signs.

7. Find the area of the quadrilateral whose vertices are (1, 2), (– 2, 3), (– 3, – 4), and (4, – 5).

8. Find the area of the polygon whose vertices are (4, 1), (2, 3), (0, 4), (– 2, 3), (– 4, 1).

9. Prove that the points (1, 2), (3, 6), (– 1, – 2) are collinear.

10. Show that the area of the triangle whose vertices are (2, 6), (– 4, 3), (– 2, 7) is four times the area of the triangle formed by joining the middle points of the sides.

198. Applications to the Proof of Geometric Theorems. We shall now give a few elementary examples to show how the methods hitherto developed may be used in the proof of geometric theorems.

Example 1. Prove that the line joining the vertex of any right triangle to the mid-point of the hypotenuse is equal to half the hypotenuse.

Let ABC be any right triangle. In order to apply the methods of coördinates we must first locate a pair of coördinate axes. Any two perpendicular lines will serve the purpose, but the work incident to the solution of many problems may usually be greatly simplified if we choose the axes judiciously. In this case it is convenient to choose the legs of the triangle as axes. The coördinates of the vertices are then (Fig. 174) (0, 0), $(a, 0)$, and $(0, b)$. The midpoint of the hypotenuse is (§ 194) $(a/2, b/2)$. The length of the line joining this point to (0, 0) is $\sqrt{(a/2)^2+(b/2)^2}=\frac{1}{2}\sqrt{a^2+b^2}$. But the length of the hypotenuse is $\sqrt{a^2+b^2}$. This proves the theorem.

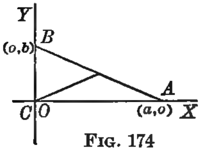


Fig. 174

Example 2. Prove that the diagonals of a parallelogram bisect each other.

Let $ABCD$ be any parallelogram. Let a side of the parallelogram lie on the x-axis a vertex being at the origin. (See Fig. 175.) We may assign the coördinates $(a, 0)$ to the vertex B, and (b, c) to the vertex D. The coördinates of C will then be $(a+b, c)$. Why?

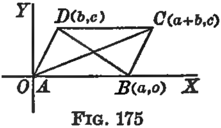


Fig. 175

We now calculate the coördinates of the mid-point of AC and also of the mid-point of BD, by the formula of § 194. It will then be seen that the midpoints coincide.

EXAMPLE 3. Prove that if the lines joining two of the vertices of a triangle to the mid-points of the opposite sides are equal, the triangle is isosceles.

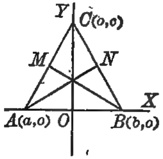


FIG. 176

Let ABC be the triangle, M, N the mid-points of the sides AC, BC, respectively, with $AN = BM$. Let the x-axis lie along the side AB and let the y-axis pass through the vertex C (Fig. 176). Let the coördinates of A, B, C be $(a, 0)$, $(b, 0)$, $(0, c)$ respectively.*

We must first state the hypothesis of the theorem analytically, *i.e.* in terms of the coördinates. To this end we note that the mid-point of AC is $M = (a/2,\ c/2)$, and that

$$\overline{BM}^2 = \left(b - \frac{a}{2}\right)^2 + \frac{c^2}{4}.$$

Similarly, we have $$\overline{AN}^2 = \left(a - \frac{b}{2}\right)^2 + \frac{c^2}{4}.$$

By hypothesis, $\overline{AN} = \overline{BM}$. Hence we have

$$\left(b - \frac{a}{2}\right)^2 + \frac{c^2}{4} = \left(a - \frac{b}{2}\right)^2 + \frac{c^2}{4}.$$

This condition gives

$$\left(b - \frac{a}{2}\right) = \pm\left(a - \frac{b}{2}\right),$$

which, when simplified, gives either $a = b$ or $a = -b$. The first result would imply that the points A and B coincide, which is contrary to the hypothesis, and is therefore rejected. The second result yields readily that $AC = BC$, which was to be proved.

EXERCISES

1. Prove analytically that the diagonals of a rectangle are equal.

2. Prove analytically that the line joining the mid-points of two sides of a triangle is half the third side.

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[*] In the figure a is a negative number. However, the discussion that follows applies at the outset to any numbers, a, b, c. It will appear later in the discussion that, under the hypothesis of the theorem, a and b must have opposite signs. One of the advantages of the analytic method is the fact that it is general, and that ordinarily special cases do not have to be considered separately.

3. Prove analytically that two triangles with the same base and equal altitudes have the same area.

4. $ABCD$ is a parallelogram, with A, C as opposite vertices. M and N are the mid-points of the sides AB and CD respectively. Prove analytically that the lines AN and CM trisect the diagonal BD.

5. If P is any point in the plane of a rectangle, prove analytically that the sum of the squares of the distances from P to two opposite vertices of the rectangle is equal to the sum of the squares of the distances from P to the other two vertices.

6. Prove analytically that, if the diagonals of a parallelogram are equal, the figure is a rectangle.

7. Prove analytically that the two straight lines which join the mid-points of the opposite sides of a quadrilateral bisect each other.

8. Show analytically that the figure formed by joining the middle points of the sides of any quadrilateral is a parallelogram.

9. If M is the mid-point of the side BC of any triangle ABC, prove that $AB^2 + AC^2 = 2(AM^2 + MC^2)$.

10. Prove analytically that the distance between the middle points of the non-parallel sides of a trapezoid is equal to half the sum of the parallel sides.

11. The difference of the squares of any two sides of a triangle is equal to the difference of the squares of their projections on the third side.

12. Prove that the sum of the squares of the sides of any quadrilateral is equal to the sum of the squares of the diagonals plus four times the square of the distance between the middle points of the diagonals.

13. If A, B, C, D are four points of a line prove the relation (due to Euler): $AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$. (The segments are directed.)

14. If M and N, respectively, are the mid-points of two segments AB and CD on the same line, show that $2\,MN = AC + BD = AD + BC$.

15. If M is the mid-point of AB and P any other point of the line AB, show that $PA \cdot PB = PM^2 - MA^2$.

16. Two sources of light of intensity α and β are situated at the points A and B respectively of a line. Find the position of a point on the line which is lighted with the same intensity by the two points. How many points satisfy the relation?

[Hint: The intensity of light at a point varies inversely as the square of the distance of the point from the source of light and directly as the intensity of the source.]

17. Two objects of weights w_1, w_2 are situated at the points A_1, A_2. The *center of gravity* of the two objects is defined to be the point of the line A_1A_2, whose simple ratio with respect to A_1, A_2 is w_2/w_1. If A_1, A_2 are on the x-axis, and their coördinates are x_1, x_2, find the coördinate of the center of gravity. Show that the center of gravity does not exist, if $w_1 = -w_2$. Give an interpretation to a negative w.

18. Given n weights $w_1, w_2, \cdots, w_n$ situated at the points $A_1, A_2, \cdots, A_n$ on a line. Find the center of gravity of A_1, A_2 with weights w_1, w_2; then the center of gravity of the point found taken with the weight $w_1 + w_2$ and A_3 with the weight w_3; then the center of gravity of this new point taken with the weight $w_1 + w_2 + w_3$ and A_4 with the weight w_4; and so on. Show that when all the n points have been used, there is obtained a point which is independent of the order in which the points were taken. The point thus determined is called the *center of gravity* of the n points. When does no center of gravity exist? Under what conditions is it indeterminate? Show that if the latter conditions hold, each of the given points is the center of gravity of the remaining ones each taken with the weight assigned to it.

19. The *first* (or *static*) *moment* of a point P of weight w about a line l is defined to be the product of w by the distance of P from l. Given n points $P_i = (x_i, y_i) (i = 1, 2, \cdots, n)$ in a plane with weights w_i, respectively, determine the coördinates of a point P of weight $w_1 + w_2 + \cdots + w_n$ such that its moment about the x-axis shall be equal to the sum of the moments about the x-axis of the points P_i and such that its moment about the y-axis shall be the sum of the moments about the y-axis of the points P_i. The point P is the center of gravity of the set of points. Compare with the result of Ex. 18.

20. The *second moment* or the *moment of inertia* of a point P with respect to a line l is defined to be the product of the weight w of P by the square of its distance from the line. Given n points P_i in a plane whose distances from a fixed line l are x_i, and whose weights are w_i respectively. Let M_1 be the sum of the first moments, M_2 the sum of the second moments of these points about the line l. Let l' be a second line, parallel to the first and h units from it (to the right or left according as h is positive or negative), and let M'_1 and M'_2 be the sum of the first and second moments of the given points about l'. Let W be the sum of the weights $w_1 + w_2 + \cdots + w_n$. Show that

$$M'_1 = M_1 - hW \text{ and } M'_2 = M_2 - 2hM_1 + h^2W.$$

199. Directed Lines and Angles. An angle from a directed line l_1 to a directed line l_2 is an angle through which l_1 must be rotated to make its direction coincide with that of l_2. Any such angle we denote by $(l_1\, l_2)$. Clearly if l_1 and l_2 intersect in a point M (Fig. 177), $(l_1\, l_2)$ is the directed angle from l_1 to l_2 as defined in § 98 since the directions of l_1 and l_2 define uniquely the half-lines issuing from M. As we observed in § 101, an angle $(l_1\, l_2)$ may have various determinations differing from each other by multiples of 360°.

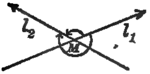

FIG. 177

The angle from the x-axis to a directed line l is called the ***inclination*** of l (Fig. 178). If the inclination of a directed line l_1 is θ_1 and the inclination of a directed line l_2 is θ_2, the angle from l_1 to l_2 is given (§ 101) by the equation

$$(6) \qquad (l_1\, l_2) = \theta_2 - \theta_1,$$

where the equality sign means *equal except possibly for multiples of 360°*.

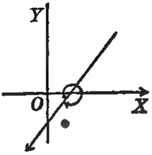

FIG. 178

200. Undirected Lines and Angles. If two lines l_1 and l_2 are not directed, an angle from l_1 to l_2, defined as an angle through which l_1 must be rotated to make it parallel to l_2, will have various determinations which differ by multiples of 180° (Fig. 179). The smallest positive (or zero) angle from l_1 to l_2 is then unique and less than 180°. The *inclination* of an undirected line is defined as the smallest positive (or zero) angle through which it is necessary to rotate the x-axis in order to make it parallel to the line. In Chapter III we used the slope m of a line to measure its inclination. It follows almost immediately from the definition of slope m and inclination θ that we have

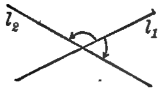

FIG. 179

$$m = \tan \theta.$$

To calculate the angle *from* a line l_1 *to* a line l_2 we make use of (6), § 199, if the inclinations θ_1, θ_2 of l_1, l_2 are known. If the slopes m_1, m_2 of l_1, l_2 are given, we find from (6), § 138

$$\tan(l_1\, l_2) = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}.$$

But $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$. Hence we have

$$\tan(l_1\, l_2) = \frac{m_2 - m_1}{1 + m_2 m_1}. \tag{7}$$

As special cases of this relation we obtain the familiar condition for parallelism and perpendicularity (§§ 64, 65). For, if the lines are parallel, $(l_1\, l_2) = 0°$ or $180°$; hence $m_1 = m_2$.

If the lines are perpendicular, $(l_1, l_2) = 90°$ or $270°$; hence

$$1 + m_1 m_2 = 0, \text{ or } m_1 = -\frac{1}{m_2}.$$

201. Standard Forms of the Equation of a Straight Line. We recall here for reference the standard forms of the equation of a straight line derived in Chapter III:

The general equation: $Ax + By + C = 0.$
The slope form: $y = mx + b.$
The point-slope form: $y - y_1 = m(x - x_1).$

The last two forms *are not general,* since they will not serve to represent lines parallel to the y-axis. The first *is general.* If the first represents a line not parallel to the y-axis ($B \neq 0$), it is readily reduced to the slope form, by solving the equation for y:

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This yields, as was shown in § 63,

$$m = -\frac{A}{B}.$$

EXERCISES

1. Construct a line through the point $(-2, 3)$ having an inclination of 60°. What is the slope? Write the equation of the line. Find the points at which the line crosses the x-axis and the y-axis.

2. Proceed as in Ex. 1 for a line passing through the point $(2, -3)$ with an inclination of 135°.

3. Find, to the nearest minute, the inclination of each of the following lines. Use a table of natural functions.

(*a*) $2x - 3y = 0.$ (*c*) $x = 2.1y + 3.5.$ (*e*) $x - y + 249 = 0.$
(*b*) $y = 0.4x + 1.7.$ (*d*) $7x + 3y - 8 = 0.$ (*f*) $x + 2y + 6 = 0.$

4. Find the tangent of the angle from the first line to the second line of each of the following pairs. Then find the angle.

(*a*) $2x - 3y = 0,$
$x + 2y + 7 = 0.$
(*b*) $5x + 2y - 10 = 0,$
$2x + 3y + 6 = 0.$
(*c*) $x + 3y - 3 = 0,$
$3x - y + 6 = 0.$
(*d*) $y = 2x + 3,$
$3x + y - 6 = 0.$

5. Find the equation of the line through $(4, 5)$ and parallel to the line joining $(-1, 2)$ and $(2, -3)$.

6. Find the equation of a line through the intersection of $2x + y - 5 = 0$ and $x - 3y + 5 = 0$, and perpendicular to the line $2x - 3y + 6 = 0$.

7. An isosceles triangle has for its base the line $x - 2y + 2 = 0$ and for its vertex the point $(-3, 5)$. The base angles are 45°. Find the equations of the other two sides and the coördinates of the other two vertices.

8. Given the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Show that they are parallel, if and only if $a_1b_2 - a_2b_1 = 0$; and that they are perpendicular, if and only if $a_1a_2 + b_1b_2 = 0$.

9. The sides of a triangle have slopes equal to $\frac{1}{2}$, 1, and 2. Show that the triangle is isosceles.

10. Find the angles of the triangle whose vertices are $(3, 4)$, $(-3, 6)$, and $(2, -1)$.

11. Find the slope of the bisector of the angle which a line of slope -2 makes with a line of slope 3.

12. The slope of a line AB is 2. Find the equation of a line through the origin which makes with AB an angle whose tangent is -1.

13. P is any point on the curve whose equation is $y^2 = 4x$. Show that the tangent to the curve at P bisects the angle which the line joining P to the point $(1, 0)$ makes with the line through P and parallel to the x-axis.

202. The Expression Ax_1+By_1+C. The expression $x-2y+3$ has the value $+2$ when $x=1$ and $y=1$; the value 0 when $x=-1$ and $y=1$; the value -2 when $x=-3$ and $y=1$. The only interpretation we are able thus far to give to these facts is that the second set of values for x and y are the coördinates of a point $(-1, 1)$ which is on the line whose equation is $x+2y+3=0$, while the other sets of values are the coördinates of points not on this line.

It seems reasonable to expect, however, that the value of the expression x_1-2y_1+3, where (x_1, y_1) is any point in the plane, must have some relation to the line whose equation is $x-2y+3=0$. This relation is indeed very simple. The reader should have no difficulty in proving that the value $+2$ obtained above from the point $(1, 1)$ represents in sign and magnitude the directed segment drawn parallel to the x-axis from the line to the point $(1, 1)$. Similarly, the value -2 represents the segment drawn parallel to the x-axis from the line to the point $(-3, 1)$.

We proceed to show that a similar result applies to the values of the left-hand member of the equation of any line in the form $Ax+By+C=0$.

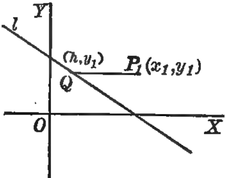

FIG. 180

Let the line l (Fig. 180) be the line whose equation is $Ax+By+C=0$, where we assume $A\neq 0$, and suppose the equation has been written so that A is *positive*. Why is this last always possible? The line is then not parallel to the x-axis. Why? Let $P_1(x_1, y_1)$ be any point in the plane and let $Q(h, y_1)$ be the point in which the line through P parallel to the x-axis meets l. Since Q is on l, we have

$$Ah+By_1+C=0,$$

or

$$By_1+C=-Ah.$$

The value of $Ax_1 + By_1 + C$, which we are seeking, is therefore equal to $Ax_1 - Ah$, or $A(x_1 - h)$. But $x_1 - h$ represents, in sign and magnitude, the segment $\overline{QP_1}$. We have then,

$$Ax_1 + By_1 + C = A \cdot \overline{QP_1}.$$

We conclude that, *if A is positive, the number $Ax_1 + By_1 + C$ is positive if (x_1, y_1) is to the right of the line $Ax + By + C = 0$, and negative if (x_1, y_1) is to the left of this line. Moreover, $Ax_1 + By_1 + C$ is proportional to the horizontal distance from the line to the point (x_1, y_1).*

Finally, if $A = 0$ and $B \neq 0$, we may suppose the equation $By + C = 0$ so written that B is positive. The line l is then parallel to the x-axis. Writing its equation in the form $y = -C/B$, it is readily seen that the expression

$$y_1 - \left(\frac{-C}{B}\right) = y_1 + \frac{C}{B}$$

represents the directed segment drawn parallel to the y-axis from the line to the point P_1 (Fig. 181). We may then conclude that, *B being positive, the number $By_1 + C$ is positive if the point (x_1, y_1) is above the line $By + C = 0$, and negative if the point (x_1, y_1) is below this line. Moreover, $By_1 + C$ is proportional to the distance of the point from the line.*

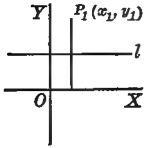

FIG. 181

By the preceding results, we may distinguish between the *positive and negative sides* of a line. If the equation of a line is written in the form $Ax + By + C = 0$ and so that its first term is positive, the right-hand side of the line is positive and the left-hand side is negative, unless the line is parallel to the x-axis. In the latter case the upper side is positive and the lower side is negative.

203. The Distance of a Point from a Line. The results of the last article enable us to find the perpendicular distance of a point $P_1(x_1, y_1)$ from the line whose equation is $Ax + By + C = 0$. If $A \neq 0$, the required distance $d = MP_1$ (Fig. 182) is evidently equal to $\overline{QP_1} \sin \theta$, where θ is the inclination of the line. This is true whether the inclination is acute or obtuse, and whether P_1 is on the positive or negative side of the given line. Since $0° \leqq \theta < 180°$, $\sin \theta$ is necessarily positive, and $d = \overline{QP_1} \sin \theta$ will have the same sign as $\overline{QP_1}$; *i.e.* it will be positive when P_1 is on the positive side of the line, and negative when P_1 is on the negative side.

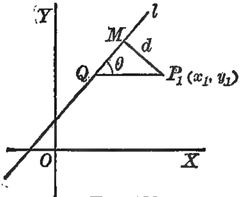

FIG. 182

We have, from the preceding article,

$$QP_1 = \frac{Ax_1 + By_1 + C}{A},$$

and, since $\tan \theta = -A/B$, we have

$$\sin \theta = \frac{A}{\sqrt{A^2 + B^2}}.$$

Hence, the required distance is

$$\textbf{(8)} \qquad \boldsymbol{MP_1 = d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}}.$$

If $A = 0$, the required distance, by § 202, is simply

$$d = y_1 + \frac{C}{B} = \frac{By_1 + C}{B}.$$

But this is precisely what (8) becomes for $A = 0$. Hence (8) is true in every case.

The distance d is positive if (x_1, y_1) is on the positive side of the line, and negative if (x_1, y_1) is on the negative side, provided the equation is written in the standard form with the first term positive.

EXAMPLE 1. To find the distance from the line $2x - 5y - 10 = 0$ to the point $(-3, 1)$. Since the equation is in standard form the desired result is obtained by substituting the coördinates of the given point in the left-hand member of the equation and dividing by the square root of the sum of the squares of the coefficients of x and y. Hence the distance d is

$$d = \frac{2(-3) - 5 \cdot 1 - 10}{\sqrt{2^2 + (-5)^2}} = \frac{-21}{\sqrt{29}}.$$

The negative sign indicates that the point is at the left of the line.

EXAMPLE 2. Find the equation of the bisector of the acute angle between the lines $3x - 4y + 12 = 0$ and $4x - 3y + 6 = 0$.

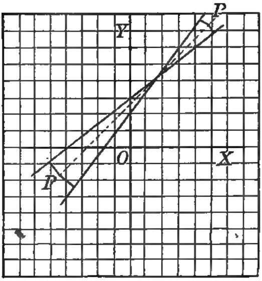

FIG. 183

First draw the lines (Fig. 183). We know from geometry that the bisector of an angle is the locus of the points equidistant from the sides of the angle. Let (x, y) be any point on the desired bisector. Inspection of the figure shows that (x, y) is on the positive side of one of the lines and on the negative side of the other. Hence, any point on the desired bisector must satisfy the condition that its distance from one of the lines is equal to minus its distance from the other. This condition is expressed by the equation:

$$\frac{3x - 4y + 12}{5} = -\frac{4x - 3y + 6}{5}, \tag{9}$$

or

$$7x - 7y + 18 = 0. \tag{10}$$

Moreover, any point which satisfies relation (9) is a point of the bisector. Hence, we conclude that the equation $7x - 7y + 18 = 0$ is the required equation.

NOTE. Had the equation of the bisector of the obtuse angle been desired the figure shows that *in this case* a point on the bisector is either on the positive side of both lines or on the negative side of both lines. Hence, any such point must satisfy the relation obtained by placing its distance from one line equal to its distance from the other line. The equation of this bisector is $x + y + 6 = 0$.

Example 3. Prove that the locus of a point which moves so that the algebraic sum of its distances from any number of fixed lines is constant, is a straight line.

Each of the given straight lines has an equation of the form $ax + by + c = 0$. The distance of any point (x, y) from such a line is

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}}.$$

The equation of the required locus is, therefore, of the form

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} + \cdots + \frac{a_nx + b_ny + c_n}{\sqrt{a_n^2 + b_n^2}} = 0.$$

Since this is an equation of the first degree, the locus is a straight line.

EXERCISES

1. Without using a figure determine whether the following points are at the right or the left of the line $2x + 3y - 5 = 0$: $(1, 2)$, $(1, -1)$, $(-2, 1)$, $(1, 1)$, $(4, -2)$, $(7, -2)$, $(4, -1)$. Then, draw a figure containing the line and the points and verify the results obtained.

2. Find the distance of the point $(3, -2)$ from the line $4x - 3y + 6 = 0$.

3. Find the distance of each of the following points from the line associated with it. In each case interpret the sign of the result.

(*a*) $(2, 5)$, $4x + 3y - 2 = 0$.
(*b*) $(-3, 7)$, $5x + 12y + 24 = 0$.
(*c*) $(2, -2)$, $3x - 4y = 0$.
(*d*) $(5, 2)$, $2x + 5 = 0$.
(*e*) $(-4, 1)$, $3y - 2 = 0$.
(*f*) (a, a), $x + y - a = 0$.
(*g*) (b, a), $ax + by = 0$.
(*h*) $(1, 3)$, $y = 2x + 5$.

4. Determine the region of the plane defined by each of the following sets of relations,

(*a*) $x + 2y + 4 > 0$, $x - 2y - 6 > 0$.
(*b*) $2x - y + 2 > 0$, $y - 2 < 0$.
(*c*) $2x - 3y + 6 > 0$, $3x + 2y - 12 < 0$, $x - y - 1 < 0$.

5. Define by inequalities (as in Ex. 4) the inside of the triangle whose sides are given by the expressions in Ex. 4, (*c*) equated to zero.

6. Define by means of inequalities the inside of the triangle whose vertices are $(-2, 5)$, $(4, 1)$, $(-1, 1)$.

7. Find the distance between the two parallel lines $3x - 6y + 5 = 0$ and $3x - 6y - 2 = 0$.

8. Find the equation of the bisector of the acute angle between the lines $2x + 3y - 4 = 0$, $x - 2y + 7 = 0$.

9. Find the equation of the bisector of the obtuse angle between the lines in Ex. 8.

10. Prove that the bisectors of the angles formed by the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular to each other.

11. Find the lengths of the altitudes of the triangle whose vertices are $(1, 2)$, $(-2, 3)$, and $(-3, -4)$.

12. Find the area of the triangle in Ex. 11 by multiplying half the length of one of the sides by the corresponding altitude, and check the result by finding the area by the formula of § 196.

13. Find the distance of the point $(1, 2)$ from the line $3x + 4y + 12 = 0$ by finding the coördinates of the foot of the perpendicular dropped from the point on the line and then using the formula for the distance between two points. Check by means of § 203.

14. If the equations of two parallel lines are $ax + by + c = 0$ and $ax + by + c' = 0$, prove that the distance between them is the absolute value of $(c - c')/\sqrt{a^2 + b^2}$.

15. Prove that the bisectors of the angles of a triangle meet in a point.

[HINT: Choose a convenient relation between the triangle and the axes.]

16. Find the altitudes of the triangle formed by the lines

$$x + 2y - 3 = 0,\ x - y = 0,\ 4x - y - 1 = 0.$$

17. Prove that the altitudes on the legs of an isosceles triangle are equal.

18. Prove that the three altitudes of an equilateral triangle are equal.

19. Prove that the sum of the absolute distances of any point within an equilateral triangle from the sides of the triangle is constant.

204. Two Equations representing the same Line. If of two equations of the first degree one can be obtained from the other by multiplying the latter by a constant, the equations obviously represent the same line, since all the points which satisfy one equation must then satisfy the other also. We now proceed to prove the converse of this statement:

If the equations $Ax + By + C = 0$ *and* $A'x + B'y + C' = 0$ *represent the same line, either one can be obtained from the other by multiplication by a constant.*

Let us suppose first that none of the numbers A, A', B, B', C, C' is zero. The intercepts of the two lines on the x-axis are then $-C/A$ and $-C'/A'$, on the y-axis $-C/B$ and $-C'/B'$. Since the lines are by hypothesis identical, we have

$$\frac{A}{C}=\frac{A'}{C'} \text{ and } \frac{B}{C}=\frac{B'}{C'}.$$

From these relations follow at once

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}=k,$$

where k is a constant. It follows that

$$A=kA',\ \ B=kB',\ \ C=kC'.$$

If C (or C') is zero, the corresponding line passes through the origin, and hence the other line must also pass through the origin; hence C' (or C) is also zero. We leave the rest of the proof as an exercise, with the suggestion that the slopes of the two lines be compared.

205. The Intercept Form. Hesse's Normal Form. We have called attention thus far to three forms of the equation of a straight line: (1) the general equation; (2) the slope form; (3) the point-slope form. Two other forms are sometimes of great convenience. These are the so-called *intercept form* and *normal form*. The ***intercept form*** is

$$\frac{x}{a}+\frac{y}{b}=1, \quad (ab\neq 0) \tag{11}$$

where a and b represent, respectively, the x- and y-intercepts of the line. This equation may be derived by finding the equation of the line through the points $(a, 0)$ and $(0, b)$. The derivation is left as an exercise. (See Ex. 21, p. 89.) This form is not applicable if the straight line passes through the origin, or if it is parallel to either axis. Why?

The ***normal form*** is associated with the name of Hesse,* who used it extensively. It uses the length p of the perpendicular dropped from the origin upon the line and the angle α which this perpendicular makes with the x-axis to determine the line.

To derive the equation when p and α are given, we try to find a relation which is satisfied by the coördinates (x, y) of every point P on the line and which is not satisfied by the coördinates of any other point. To this end (Fig. 184) we note that the projection of the broken line OMP on the perpendicular OQ is equal to p, if and only if P is on the line. The projections of the parts OM and MP on OQ are, respectively, $x \cos \alpha$ and $y \sin \alpha$. The desired equation is, therefore,

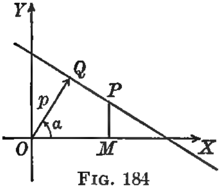

FIG. 184

$$x \cos \alpha + y \sin \alpha = p. \tag{12}$$

We shall take the positive direction of OQ, or p, from the origin towards the line, and choose the positive angle XOQ to be α. It is then evident that the position of any line is determined by a pair of values of p and α, it being understood that p and α are positive and that α is less than 360°.

Moreover every line determines a single positive value of p and a single positive angle α less than 360°, unless $p = 0$. When $p = 0$ the line evidently passes through the origin and the above rule for the positive direction of p becomes meaningless. When $p = 0$, it is customary to choose $\alpha < 180°$.

To reduce the general equation $Ax + By + C = 0$ to the normal form, we need merely observe that in the latter form an essential condition is that the coefficients of x and y are numbers the sum of whose squares is 1, since $\sin^2 \alpha + \cos^2 \alpha = 1$. We must then multiply all the coefficients of $Ax + By + C = 0$ by a number k, so chosen that $(kA)^2 + (kB)^2 = 1$. This condition will be satisfied if

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* LUDWIG OTTO HESSE (1811–1874), a noted German mathematician.

$$k = \pm \frac{1}{\sqrt{A^2 + B^2}}.$$

Therefore the desired reduction is obtained by dividing the equation through by $\pm\sqrt{A^2 + B^2}$, and transposing the constant term to the right-hand side of the equation:

$$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}}y = \frac{-C}{\pm\sqrt{A^2+B^2}}.$$

The sign of the radical must be chosen opposite to the sign of C, or if $C = 0$, the same as that of B. Why?

One advantage of the normal form is that every line may have its equation written in the normal form. Whether the line passes through the origin or is parallel to an axis is immaterial.

EXERCISES

1. Reduce the following equations to the normal form. Find in each case the values of α and p.

(*a*) $4x + 3y - 10 = 0$. (*d*) $3x - 2y + 6 = 0$.
(*b*) $x - y + 5 = 0$. (*e*) $y = 2x - 3$.
(*c*) $x + \sqrt{3}y = 0$. (*f*) $x = 2y - 5$.
(*g*) The equation of the line whose intercepts are -5 and 2, respectively.

2. Reduce to the intercept form each of the lines in Ex. 1 for which such reduction is possible.

3. What are the normal forms of the equations $x = 3$, $2x + 3 = 0$, $y - 1 = 0$?

4. Derive the process of reducing the equation $Ax + By + C = 0$ to the normal form by using the fact (derived from § 203) that $p = -C/\sqrt{A^2+B^2}$.

5. What system of lines is obtained from the normal form, if α has a fixed value, while p is allowed to assume different values? If p has a fixed value and α is allowed to assume different values?

6. Find the equations of the lines which pass through the point (1, 2) and are two units distant from the origin.

7. Find the equations of the lines parallel to $5x + 12y = 13$ and 3 units distance from it.

8. Find the equations of the lines parallel to $3x + 4y = 13$ and 7 units distance from it.

MISCELLANEOUS EXERCISES

1. Find the equation of the straight line passing through the point (3, 4), such that the segment of the line between the axes is bisected at that point.

2. Show that the lines $y = ax + a$, for all values of a, pass through a fixed point.

3. Given $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$, the equations of three lines forming a triangle. Show that the equation of any line $Ax + By + C = 0$ in the plane may be written in the form

$$k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) + k_3(a_3x + b_3y + c_3) = 0,$$

where k_1, k_2, k_3 are constants.

4. Find the ratio in which the line $3y = 6 - x$ divides the segment joining the points (6, 1) and (– 3, 2).

5. Find the equation of the line that passes through the point (1, 7) and makes an angle of 45° with the line $x + 2y = 1$.

6. Find the equation of the line that passes through the point (1, 7) and makes an angle of – 45° with the line $x + 2y = 1$.

7. Prove analytically that the perpendicular bisectors of the sides of a triangle meet in a point.

8. Prove analytically that the altitudes of a triangle meet in a point.

9. Prove analytically that the bisectors of the interior angles of a triangle meet in a point.

10. Prove analytically that the bisectors of two exterior angles of a triangle and of the third interior angle meet in a point.

11. The equations of two sides of a parallelogram are $x - 2y = 1$, $x + y = 3$. Find the equations of the other two sides if one vertex is at (0, – 1).

12. Find the equation of the line passing through the point (1, 1) and dividing the segment from (– 7, – 2) to (7, – 1) in the ratio 2 : 5.

13. Two vertices of an equilateral triangle are (1, 1) and (4, 1). Find the coördinates of the third vertex. There are two solutions.

14. Find the equation of the line passing through the point (1, 2) and intersecting the line $x + y = 4$ at a distance $\frac{1}{2}\sqrt{10}$ from this point.

15. Find the equation of the line through the point (1, 2) which forms the base of an isosceles triangle with the sides $2x - y = 1$, $x + y = 1$.

16. A straight line moves so that the sum of the reciprocals of its intercepts on the two axes is constant. Show that the line passes through a fixed point.

17. If a straight line be such that the sum of the perpendiculars upon it from any number of fixed points is zero, show that it will pass through a fixed point.

18. Find the equations of the sides of the square of which two opposite vertices are $(3, -4)$ and $(1, 1)$.

19. Derive the formula for the distance of a point (x_1, y_1) from the line $Ax + By + C = 0$ by finding the intersection of the perpendicular through the given point and the given line, and then using the formula for the distance between two points.

20. Prove that if the sum of the first moments of n points with respect to each of two given perpendicular lines is zero, the sum of the moments of these points with respect to any line in the plane through the intersection of the given lines is zero. (See Ex. 19, p. 305.)

[HINT: Take the given perpendicular lines to be the axes of coördinates.]

21. If with the center of gravity of n points in a plane is associated the sum of the weights of the n points, prove that the sum of the first moments of the n points with respect to any line in the plane is equal to the first moment of the center of gravity with respect to the same line.

22. Given two half-lines r, s issuing from a point P, a third half-line t through P is completely determined if the ratio $\sin(rt)/\sin(ts) = k$ is known. The ratio k is called the *simple ratio* of t with respect to r, s. Prove that the equations $l = 0$ and $m = 0$ of r and s, respectively, may be so written that, for all positions of t, the equation of t is $l - km = 0$.

23. Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ and a straight line $ax + by + c = 0$ which meets the line P_1P_2 in Q. Find the simple ratio P_1Q/QP_2.

[HINT: This can be obtained very readily from a figure by observing the relation between the desired ratio and the ratio of the distances of P_1, P_2 from the given line.]

24. From the last exercise derive the theorem of Menelaus: *If a straight line cuts the sides of a triangle ABC in three points A', B', C', the product of simple ratios*

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A}$$

is -1. The point A' is on the side opposite A, B' on the side opposite B, C' on the side opposite C.

CHAPTER XII

THE CIRCLE

206. Review. The circle is the locus of a point which moves so that its distance from a fixed point, called the *center*, is constant. This constant distance is called the *radius* of the circle.

If the center of a circle is at the point (h, k) and the radius is r, the equation of the circle is

$$(1) \qquad (x-h)^2+(y-k)^2=r^2.$$

For, this equation expresses directly the fact that the square of the distance from the given point (h, k) to the variable point (x, y) is r^2. Hence, every point on the circle satisfies this equation and, conversely, any point not on the circle does not satisfy it.

In particular, if the center is at the origin $(h=0, k=0)$, the equation becomes

$$(2) \qquad x^2+y^2=r^2.$$

We note also that equation (1) when expanded has the form

$$(3) \qquad x^2+y^2+Dx+Ey+C=0.$$

It follows that every circle in the plane may be represented by an equation of this form. To what extent is the converse true? Under what conditions does an equation of the form (3) represent a circle? The answer to this question may be obtained by reference to the method of § 183.

We desire to complete the square on the terms in x, and also on the terms in y. Therefore we rewrite the equation in the form

$$(x^2+Dx+\quad)+(y^2+Ey+\quad)=-C.$$

To complete the squares in the two parentheses we need to add $D^2/4$ to the first and $E^2/4$ to the second; to maintain the validity of the equation we must add the same terms to the right-hand member. We then obtain

$$\left(x^2 + Dx + \frac{D^2}{4}\right) + \left(y^2 + Ey + \frac{E^2}{4}\right) = -C + \frac{D^2}{4} + \frac{E^2}{4},$$

or

$$\left(x + \frac{D}{2}\right)^2 + \left(y + \frac{E}{2}\right)^2 = \frac{D^2 + E^2 - 4C}{4}.$$

Since the sum of the squares of two real numbers is positive or zero, the left-hand member is positive or zero if x, y, D, E are real numbers. Hence the equation can be satisfied by real coördinates x, y only if $D^2 + E^2 - 4C$ is a positive number or zero.

If $D^2 + E^2 - 4C$ is positive, equation (3) represents a circle with center at $(-D/2, -E/2)$ and radius equal to

$$\tfrac{1}{2}\sqrt{D^2 + E^2 - 4C}.$$

If $D^2 + E^2 - 4C$ is zero, equation (3) is satisfied by the coördinates of the point $(-D/2, -E/2)$ and by the coördinates of no other (real) point.

If $D^2 + E^2 - 4C$ is negative, equation (3) represents no real locus. The answer to our question may then be formulated as follows: If (3) represents a curve at all, it represents a circle.

207. The Equation of a Circle satisfying given Conditions. The problem of finding the equation of a circle satisfying given conditions resolves itself simply into the problem of determining from the given conditions the values of h, k, r in equation (1), or of D, E, C in equation (3) of § 206. The following examples will illustrate the methods that may be used:

Example 1. *Find the equation of the circle passing through the three points* $(3, -5)$, $(3, 1)$, *and* $(4, 0)$.

The desired equation must be of the form (3), and must be satisfied by the coördinates of each of the three given points. If the first point satisfies this equation, D, E, and C must be such that

$$3^2+(-5)^2+D\cdot 3+E(-5)+C=0,$$

i.e. such that

$$3D-5E+C=-34.$$

We find similarly from the second and third of the given points,

$$3D+E+C=-10,$$
$$4D+C=-16.$$

Solving these three linear equations for D, E, C, we obtain

$$D=-2,\ E=4,\ C=-8.$$

The desired equation is, therefore,

$$x^2+y^2-2x+4y-8=0.$$

Another method of solving this problem would be to regard (h, k) as unknown coördinates of the center. They must satisfy the two equations

$$(3-h)^2+(-5-k)^2=(3-h)^2+(1-k)^2,$$
$$(4-h)^2+(0-k)^2=(3-h)^2+(1-k)^2.\quad \text{(Why?)}$$

By solving these equations we can determine h and k. Having found the center, it is easy to determine the radius. Then the desired equation can be written down in form (1). The completion of the work here suggested is left as an exercise. What other method could be used to solve this problem?

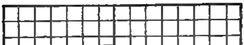


FIG. 185

EXAMPLE 2. *Find the equation of the circle inscribed in the triangle whose sides are* $y-3=0$, $3x-4y-9=0$, and $12x+5y+9=0$.

Let (h, k) be the center of the circle. It must be equidistant from the three sides. The distances of (h, k) from the three given lines are $-(k-3)$, $-\frac{1}{5}(3h-4k-9)$, and $\frac{1}{13}(12h+5k+9)$, the signs being so chosen that each of these numbers is positive when (h, k) is within the triangle. (See Fig. 185.)

By placing the first of these distances equal to the second and third, respectively, we obtain two equations involving h and k. The solution of these two equations yields $h=1$, $k=1$. Hence the center is the point (1, 1). The radius is evidently equal to 2. Why? Therefore the required equation is

$$(x-1)^2+(y-1)^2=4, \text{ or } x^2+y^2-2x-2y-2=0.$$

EXERCISES

1. Write the equations of the circles described below:

(*a*) Center at the origin, radius equal to 5.

(*b*) Center at (1, 2), radius $= 4$.

(*c*) Center at $(-3, -2)$, radius $= 3$.

(*d*) Center at (a, a) and radius $= a$.

(*e*) Center at $(-2, 1)$ and passing through the point $(3, -2)$.

(*f*) Center at (2, 1) and tangent to the x-axis.

2. Discuss fully the locus of each of the following equations:

(*a*) $x^2 + y^2 - 2x + 4y + 1 = 0$.
(*d*) $x^2 + y^2 + 1 = 0$.
(*b*) $x^2 + y^2 - 4x - 6y = 0$.
(*e*) $x^2 + y^2 + 2x - 6y + 10 = 0$.
(*c*) $x^2 + y^2 + 3x - 4 = 0$.
(*f*) $x^2 + y^2 + 2ax + 2a^2 = 0$.
(*g*) $3x^2 + 3y^2 + 2x - 4y - 8 = 0$.

3. What can be said of the coefficients D, E, and C in the general equation if the equation represents a circle which

(*a*) passes through the origin?

(*b*) has its center on the x-axis? on the y-axis?

(*c*) has its center on the line $x + y = 0$?

(*d*) touches both axes?

(*e*) has its radius equal to 2?

4. Find the equations of the circles described below:

(*a*) Passing through the points (0, 2), (1, 4), (1, 0).

(*b*) Circumscribing the triangle whose sides are the lines $x + y - 3 = 0$, $x - 2y + 6 = 0$, $x + 2 = 0$.

(*c*) Inscribed in the triangle whose vertices are (0, 2), $(0, -4)$, and $(-4, 1)$.

(*d*) Having $(-2, 4)$ and $(4, -2)$ as the extremities of a diameter.

(*e*) Passing through the points (1, 2) and (2, 1) and having its center on the line $2x + y + 2 = 0$.

(*f*) Tangent to both coördinate axes and passing through the point (2, 1). How many solutions are there?

5. Prove analytically that any angle inscribed in a semicircle is a right angle.

6. Prove that the locus of a point which moves so that the sum of the squares of its distances from any number of fixed points is constant is a circle. Find the coördinates of the center of this circle in terms of the coördinates of the fixed points. If the number of fixed points is three, how is the center of the circle related to the triangle whose vertices are at the fixed points?

7. Find the equation of the locus of a point which moves so that the ratio of its distances from two fixed points is constant and equal to k. Determine fully this locus. Examine especially the case $k = 1$.

[HINT: Let the two fixed points be $(a, 0)$ and $(-a, 0)$].

8. Draw the loci of Ex. 7 for different values of k. Prove that if any one of these loci crosses the line joining the two given points in P and Q, respectively, and the mid-point of the segment joining the given points is M, we have $MP \cdot MQ$ equal to the square of half the segment.

208. Tangent to a Circle. Point Form. In § 184 we saw how the slope of the curve $Ax^2 + By^2 + Dx + Ey + C = 0$ at any point (x_1, y_1) *on the curve* could be derived. Applying this method to the circle

$$x^2 + y^2 + Dx + Ey + C = 0,$$

we find the slope m at (x_1, y_1) on the curve to be

$$m = -\frac{2x_1 + D}{2y_1 + E}.$$

The equation of the tangent at the point (x_1, y_1) is, therefore,

$$y - y_1 = -\frac{2x_1 + D}{2y_1 + E}(x - x_1).$$

Simplifying, we obtain

$$(4) \quad 2x_1x + 2y_1y + Dx + Ey - 2x_1^2 - 2y_1^2 - Dx_1 - Ey_1 = 0.$$

But (x_1, y_1) is on the curve, and hence

$$x_1^2 + y_1^2 + Dx_1 + Ey_1 + C = 0.$$

If this identity be multiplied by 2 and added to (4) we obtain

$$2x_1x + 2y_1y + Dx + Ey + Dx_1 + Ey_1 + C = 0,$$

or

$$(5) \qquad x_1x + y_1y + \tfrac{1}{2}D(x + x_1) + \tfrac{1}{2}E(y + y_1) + C = 0.$$

As a special case of this equation (for $D = 0$, $E = 0$, $C = -r^2$) we obtain the equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) to be

$$(6) \qquad x_1x + y_1y = r^2.$$

209. Tangent to a Circle. Slope Form. Another form of the equation of a tangent to the circle $x^2+y^2=r^2$ is often very serviceable. It is derived as follows. The straight line $y=mx+b$ meets the circle $x^2+y^2=r^2$ in points whose abscissas are given by the equation

$$x^2+(mx+b)^2=r^2.$$

When expanded this equation becomes

$$(1+m^2)x^2+2\,mbx+b^2-r^2=0.$$

The roots of this equation will be real and distinct, real and coincident, or imaginary, according as

$$4\,m^2b^2-4(1+m^2)(b^2-r^2)$$

is positive, zero, or negative.

Translated into geometric terms, this means that the line $y=mx+b$ will meet the circle in two distinct points, two coincident points, or not at all, according as the expression above is positive, zero, or negative. If the line meets the circle in two coincident points, the line is a tangent. The condition

$$4\,m^2b^2-4(1+m^2)(b^2-r^2)=0$$

yields, after simplification,

$$b^2=(1+m^2)\,r^2,$$

or,

$$b=\pm\,r\sqrt{1+m^2}.$$

Hence, *for all values of m the equation*

(7) $$y=mx\pm r\sqrt{1+m^2}$$

represents a tangent to the circle $x^2+y^2=r^2$.

It follows at once that *for all values of m the equation*

$$y-k=m(x-h)\pm r\sqrt{1+m^2}$$

represents a tangent to the circle $(x-h)^2+(y-k)^2=r^2$.

EXERCISES

1. Write the equations of the tangents to the following circles at the points indicated:

(*a*) $x^2 + y^2 = 25$, at $(3, -4)$.
(*b*) $x^2 + y^2 = 5$, at $(-1, 2)$.
(*c*) $x^2 + y^2 = 4$, at $(0, 2)$.
(*d*) $x^2 + y^2 = 13$, at the points where $x = 3$.
(*e*) $x^2 + y^2 = 10$, at the points where $y = 1$.
(*f*) $x^2 + y^2 + 2x - 4y = 0$, at $(1, 1)$.

2. Derive the equation of the tangent to the circle $(x - h)^2 + (y - k)^2 = r^2$ at the point (x_1, y_1) by making use of the fact that the tangent is perpendicular to the radius through the point of contact.

3. Find the intersections of the following circles with the lines indicated:

(*a*) $x^2 + y^2 = 5$ and $y = 3x + 5$.
(*b*) $x^2 + y^2 = 25$ and $x - 2y - 5 = 0$.
(*c*) $x^2 + y^2 = 13$ and $3x + 2y - 13 = 0$.
(*d*) $x^2 + y^2 = 10$ and $y = 3x + 10$.
(*e*) $x^2 + y^2 = 4$, and $y = -2x + 4$, $y = -2x + 2\sqrt{5}$, $y = -2x + 5$.

Draw a careful figure showing the circle and the three lines.

4. Write the equations of the tangents to the following circles, the slopes of the tangents being as indicated. Find the points of contact.

(*a*) $x^2 + y^2 = 10$, slope $= -3$.
(*b*) $x^2 + y^2 = 5$, slope $= \frac{1}{2}$.
(*c*) $x^2 + y^2 = 13$, slope $= \frac{2}{3}$.
(*d*) $x^2 + y^2 = 25$, slope $= 0$.
(*e*) $(x-1)^2 + (y+2)^2 = 10$, slope $= 3$.

5. Will the equation $y = mx \pm r\sqrt{1 + m^2}$ represent *any* tangent to the circle $x^2 + y^2 = r^2$. Why?

6. What is the point of contact of the tangent $y = mx + r\sqrt{1 + m^2}$ to the circle $x^2 + y^2 = r^2$? From this result derive the equation $x_1x + y_1y = r^2$.

7. Any circle through the origin has an equation of the form $x^2 + y^2 + Dx + Ey = 0$. Why? Prove that the equation of the tangent at the origin is $Dx + Ey = 0$. This may be done in at least two different ways.

8. Prove analytically that from an external point two real tangents can be drawn to a circle.

9. Derive the equation $y = mx \pm r\sqrt{1 + m^2}$ directly from the property that a tangent to a circle is perpendicular to the radius through the point of contact.

210. The Value and Sign of the Expression $x_1^2+y_1^2+Dx_1+Ey_1+C$. The left-hand member of the standard equation $(x-h)^2+(y-k)^2=r^2$ represents the square of the distance from the point (x, y) to the point (h, k). *Hence the expression*

$$(8) \qquad (x_1-h)^2+(y_1-k)^2-r^2$$

is positive, negative, or zero according as (x_1, y_1) *is outside, inside, or on the circle whose equation is* $(x-h)^2+(y-k)^2=r^2$.

Moreover, from Fig. 186 it follows that if (x_1, y_1) is a point outside the circle, the expression (8) is equal to the square of the length of a tangent drawn from the point (x_1, y_1) to the circle. Since the left-hand member of the general equation $x^2+y^2+Dx+Ey+C=0$ may be written in the form $(x-h)^2+(y-k)^2-r^2$ we may conclude that *the sign of the expression* $x_1^2+y_1^2+Dx_1+Ey_1+C$ *is positive or negative according as the point* (x_1, y_1) *is outside or inside the circle* $x^2+y^2+Dx+Ey+C=0$; *and, if positive, it represents the square of the length of a tangent drawn from the point* (x_1, y_1) *to the circle.*

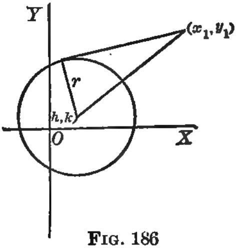


FIG. 186

211. The Equations of the Tangents from an External Point. Suppose we desire to find the equations of the tangents drawn from an external point (x_1, y_1) to the circle $x^2+y^2=r^2$. Three methods will be discussed:

EXAMPLE. *Find the equations of the tangents drawn from the point* $(4, -3)$ *to the circle* $x^2+y^2=5$.

METHOD 1. Let (x_1, y_1) be the point of contact of one of the tangents. The equation of the tangent at this point is $x_1x+y_1y=5$. However, since this tangent passes through the point $(4, -3)$ we have

$$(9) \qquad 4x_1-3y_1=5.$$

But the point (x_1, y_1) is on the circle $x^2 + y^2 = 5$. Therefore

$$x_1^2 + y_1^2 = 5. \tag{10}$$

Solving equations (9) and (10), we find the points of contact to be (2, 1) and $(-2/5, -11/5)$. Therefore the required tangents are $2x + y - 5 = 0$ and $2x + 11y + 25 = 0$.

Method 2. From § 209 it follows that any tangent (not parallel to the y-axis) to the circle $x^2 + y^2 = 5$ is of the form $y = mx \pm \sqrt{5}\sqrt{1 + m^2}$. Since this tangent is to pass through the point $(4, -3)$ we have

$$-3 = 4m \pm \sqrt{5}\sqrt{1 + m^2},$$

which simplifies to $11m^2 + 24m + 4 = 0$; this gives $m = -2$, or $-2/11$. Substituting these values in $y = mx \pm \sqrt{5}\sqrt{1 + m^2}$ and simplifying we have $2x + y - 5 = 0$ and $2x + 11y + 25 = 0$.

Method 3. The equation of any line through the point $(4, -3)$ is of the form $y + 3 = m(x - 4)$. Eliminating y between this equation and $x^2 + y^2 = 5$ we have

$$(m^2 + 1)x^2 + x(-8m^2 - 6m) + (16m^2 + 24m + 4) = 0. \tag{11}$$

Now since we desire $y + 3 = m(x - 4)$ to be tangent, equation (11) must have equal roots, *i.e.* $(-8m^2 - 6m)^2 - 4(m^2 + 1)(16m^2 + 24m + 4) = 0$ or $11m^2 + 24m + 4 = 0$ which gives $m = -2$, or $-2/11$. Therefore the equations of the tangents are $2x + y - 5 = 0$ and $2x + 11y + 25 = 0$.

212. The Polar of a Point with respect to a Circle.

Let us apply the first method of § 211 for finding the equations of the tangents from an external point to a circle, to the general problem of finding the equations of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$. The coördinates (x', y') of the point of contact are then found by solving simultaneously the pair of equations $x'x_1 + y'y_1 = r^2$, $x'^2 + y'^2 = r^2$. The first equation expresses the fact that the point (x_1, y_1) is on the tangent $x'x + y'y = r^2$; the second, that (x', y') is on the circle.

This shows that the straight line $x_1x + y_1y = r^2$, where (x_1, y_1) is any external point, meets the circle in the points of contact of the tangents drawn from (x_1, y_1). In other words,

$$x_1x + y_1y = r^2 \tag{12}$$

is the equation of the line joining the points of contact of the tangents through (x_1, y_1), if the latter point is outside the circle. If this point is on the circle, we know that (12) is the equation of the tangent at the given point. Finally, if (x_1, y_1) is inside the circle, (12) represents a definite straight line determined by the point and the circle. This straight line (12), whether (x_1, y_1) is outside, on, or inside the circle, is called the ***polar*** of (x_1, y_1) with respect to the circle. The polar of (x_1, y_1) with respect to a circle is then a uniquely determined line for every point (x_1, y_1) in the plane, except the center of the circle. (See Fig. 187.) Why this exception?

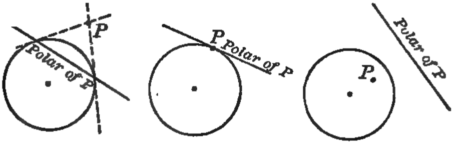


FIG. 187

EXERCISES

1. Are the following points inside, outside, or on the circle $x^2 + y^2 - 2x + 6y - 15 = 0$? (1, 2), (1, 0), (1, 4), (– 3, 0), (3, 0), (0, 2), (5, 1). For the points outside, find the length of the tangents drawn to the circle. Draw carefully a figure to illustrate each of your results.

2. What is the length of the tangents drawn from (1, 1) to the circle whose equation is $2x^2 + 2y^2 + 3x - 5y - 1 = 0$?

[CAUTION: The equation is not in the standard form.]

3. Find the equations of the tangents drawn from the following points to the circle indicated:

(*a*) $(-2, 4)$; $x^2 + y^2 = 10$.
(*b*) $(5, -1)$; $x^2 + y^2 = 13$.
(*c*) $(3, -1)$; $x^2 + y^2 = 2$.
(*d*) $(3, 2)$; $x^2 + y^2 = 4$.
(*e*) $(4, 3)$; $x^2 + y^2 = 16$.
(*f*) $(7, 1)$; $x^2 + y^2 = 25$.

4. Find the equations of the tangents drawn from (0, 4) to the circle $x^2 + y^2 - 2x + 6y - 15 = 0$.

5. Show that the polar of a point P with respect to a circle is perpendicular to the radius or radius extended through the point P.

6. Show that if P is inside the circle, the polar of P is wholly outside the circle.

7. Show that if the polar of P with respect to a circle whose center is O cuts the line OP in Q, then $OP \cdot OQ = r^2$, where r is the radius of the circle.

[HINT: Let the center O be the origin and the line OP the x-axis.]

8. Show that if the polar of a point with respect to a given circle is given, the point is uniquely determined.

[HINT: This follows directly from the results of Exs. 5 and 7; or it may be proved directly by identifying the given polar $ax + by + c = 0$ with the equation $x_1x + y_1y = r^2$. In the latter case we should have $x_1/a = y_1/b = -r^2/c$, which determines x_1, y_1 uniquely.]

9. A straight line is drawn through a given point P, cutting a given circle in the points A and B. Calculate the length of the segments PA and PB. Let P be chosen as origin and the line through P and the center of the circle as x-axis. The equation of the circle is then $x^2 + y^2 + Dx + C = 0$. If ρ is one of the segments PA or PB and α is the angle which PA makes with the x-axis, the coördinates of A or B are $(\rho \cos \alpha, \rho \sin \alpha)$. Since this point is on the circle we have the equation

$$(\rho \cos \alpha)^2 + (\rho \sin \alpha)^2 + D\rho \cos \alpha + C = 0$$

for determining the two values of ρ. This equation reduces to

$$\rho^2 + D \cos \alpha \cdot \rho + C = 0.$$

It may be noted that the product of the roots $\rho_1\rho_2$ of this equation is C, *i.e.* independent of α. What theorem of elementary geometry does this prove? Prove also that the product $PA \cdot PB$ is positive or negative according as P is outside or inside the circle.

213. The Intersection of Two Circles.

Given two circles

$$x^2 + y^2 + D_1x + E_1y + C_1 = 0,$$

and

$$x^2 + y^2 + D_2x + E_2y + C_2 = 0.$$

The coördinates of the points of intersection are found by solving the equations simultaneously. Subtracting the equations, we have

$$(D_1 - D_2)x + (E_1 - E_2)y + C_1 - C_2 = 0.$$

Every point common to the two circles will satisfy this last equation, which is the equation of a straight line. Therefore the problem of finding the points of intersection of two circles is equivalent algebraically to that of finding the intersections of a straight line and a circle. This problem leads essentially to the solution of a quadratic equation in one unknown. Therefore we may conclude that two circles may intersect in two distinct points (two real roots), may be tangent to each other (coincident roots), or may not intersect at all (imaginary roots).*

214. Orthogonal Circles. Two circles which intersect at right angles are said to be ***orthogonal.*** In this case the tangents to the two circles at a point of intersection must be perpendicular, and the two tangents pass respectively through the centers of the circles (Fig. 188). The condition for orthogonality is then simply that the sum of the squares of the radii of the circles shall be equal to the square of the distance between their centers. If the centers are $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$ and the radii are r_1 and r_2 respectively, the condition for orthogonality is

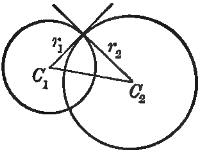


FIG. 188

$$r_1^2 + r_2^2 = (h_1 - h_2)^2 + (k_1 - k_2)^2.$$

If the equations of the circles are

$$(13) \qquad \begin{aligned} x^2 + y^2 + D_1x + E_1y + C_1 = 0, \\ x^2 + y^2 + D_2x + E_2y + C_2 = 0, \end{aligned}$$

this condition becomes (see § 206)

$$\frac{D_1^2 + E_1^2 - 4\,C_1}{4} + \frac{D_2^2 + E_2^2 - 4\,C_2}{4} = \frac{(D_1 - D_2)^2}{4} + \frac{(E_1 - E_2)^2}{4},$$

which when simplified gives

$$D_1D_2 + E_1E_2 - 2(C_1 + C_2) = 0.$$

---

* The reasoning above breaks down, if $D_1 - D_2 = 0$ and $E_1 - E_2 = 0$, that is when the circles are concentric. In this case, unless $C_1 - C_2 = 0$ also (in which case the two circles coincide), the two equations are inconsistent and have no common solution, real or imaginary.

215. Pencil of Circles. Let the left-hand members of the equations (13), § 214, be represented by M_1 and M_2 respectively. Let us consider the locus of the equation

$$M_1 - kM_2 = 0, \tag{14}$$

where k is an arbitrary constant. This equation may, if $k \neq 1$, be written in the form

$$x^2 + y^2 + \frac{D_1 - kD_2}{1-k}x + \frac{E_1 - kE_2}{1-k}y + \frac{C_1 - kC_2}{1-k} = 0, \tag{15}$$

which represents a circle for each value of $k (\neq 1)$. When $k = 1$, equation (14) represents the straight line

$$(D_1 - D_2)x + (E_1 - E_2)y + C_1 - C_2 = 0. \tag{16}$$

The system of circles obtained by giving different values to k, is called the ***pencil of circles*** determined by the two given circles. The straight line (16) is called the ***radical axis*** of the two given circles, and of the pencil.

The following properties of a pencil of circles are readily proved:

If the two given circles intersect in two points A and B, every circle of the pencil passes through A and B.

If the two given circles are tangent to each other at a point A, all the circles of the pencil are tangent at A.

Through any point in the plane not on the radical axis of the circles passes one and only one circle of the pencil. The proofs of these theorems are left as exercises.

Further properties of pencils of circles will be found in the following exercises.

EXERCISES

1. Find the coördinates of the points of intersection of the following pairs of circles:

(*a*) $x^2 + y^2 = 5$ and $x^2 + y^2 + 2x - 4y + 1 = 0$.
(*b*) $x^2 + y^2 - x + 2y = 0$ and $x^2 + y^2 + 2x - 4y = 0$.
(*c*) $x^2 + y^2 + 2x - 17 = 0$ and $x^2 + y^2 - 13 = 0$.

2. Write the equation of the radical axis of each pair of circles given in Ex. 1.

3. Prove that the tangents drawn from any point of the radical axis of two circles to the two circles are equal.

4. Prove that the circles $x^2 + y^2 + 6x - 2y + 2 = 0$ and $x^2 + y^2 + 4y + 2 = 0$ are tangent to each other. Find their point of contact and the equation of their common tangent.

5. Find the equation of the circle through the intersections of the circles $x^2 + y^2 - 4x - 4 = 0$ and $x^2 + y^2 + 2x - 6y - 2 = 0$ and the point $(3, 3)$. [It is not necessary to find the intersections.]

6. Prove that the following circles are orthogonal: $x^2+y^2-2x-4=0$ and $x^2+y^2-6y+4=0$. In general for the circles: $x^2 + y^2 + Dx - C = 0$ and $x^2 + y^2 + Ey + C = 0$.

7. Determine C so that $x^2 + y^2 - 2x + 4y - 8 = 0$ and $x^2 + y^2 + 2x + C = 0$ are orthogonal.

8. Prove that the locus of the centers of the circles of a pencil is a straight line perpendicular to the radical axis of the pencil.

9. Prove that if the radical axis of a pencil of circles is chosen as the y-axis and the line of centers as the x-axis, the equation of any circle of the pencil is of the form $x^2 + y^2 + kx + C = 0$, where C is the same for all circles of the pencil; and that all circles obtained by varying k in this equation are circles of the same pencil.

10. The circles of the pencil in Ex. 9 intersect in distinct points, are tangent to each other, or do not intersect at all, according as C is negative, zero, or positive. In case $C = 0$, all the circles of the pencil are tangent to one another at the origin. Draw carefully three figures, illustrating the three kinds kinds of pencils here indicated.

11. Find the equation of a circle which is orthogonal to two given circles of the pencil in Ex. 9.

[HINT: Let the two given circles be

$$x^2 + y^2 + k_1x + C = 0 \text{ and } x^2 + y^2 + k_2x + C = 0,$$

and let the required circle be $x^2 + y^2 + D_2x + E_2y + C_2 = 0$. If this circle is to be orthogonal to each of the given circles we must have (§ 214)

$$D_2k_1 - 2(C + C_2) = 0 \text{ and } D_2k_2 - 2(C + C_2) = 0.$$

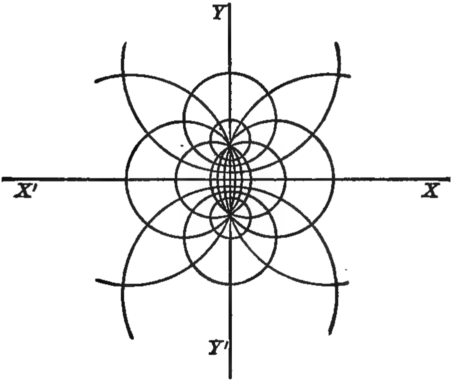
These equations give $D_2 = 0$ and $C_2 = -C$. Hence the required equation is $x^2+y^2+E_2y-C=0$. This yields two remarkable results: (1) The coefficient E_2 is undetermined, and by varying E_2 we have a pencil of circles each of which satisfies the condition of being orthogonal to the two given circles. (2) The equation found is independent of k_1, and k_2. Hence, every circle of the pencil just found is orthogonal to each of the circles of the given pencil. Writing l for E_2 to obtain uniformity of notation, we have found two pencils of circles:

$$x^2 + y^2 + kx + C = 0$$

and

$$x^2 + y^2 + ly - C = 0,$$

such that every circle of either pencil is orthogonal to each circle of the other pencil. These two pencils of circles are said to form an **orthogonal system.** (See the adjacent figure.)]



12. In an orthogonal system of circles, the centers of the circles of one pencil are on the radical axis of the other pencil.

13. If the circles of one pencil of an orthogonal system intersect in two distinct points A and B, the circles of the other system do not intersect at all, but pass between the points A and B.

14. If the circles of one pencil of an orthogonal system are mutually tangent to each other at a point A, the circles of the other pencil are also mutually tangent at A.

15. Prove that the three radical axes of three circles (not belonging to the same pencil) taken two by two intersect in a point. This point is called the ***radical center.*** Show that it is the center of a circle orthogonal to each of the three given circles and that the tangents drawn from it to the given circles are equal.

MISCELLANEOUS EXERCISES

1. Find the condition that $ax + by + c = 0$ be tangent to the circle $x^2 + y^2 = r^2$.

2. Find the equation of the circle passing through the points (0, 0), $(a, 0)$, and $(0, b)$.

3. Show that the equation of the circle having the points (x_1, y_1) and (x_2, y_2) as the extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

[HINT: The circle is the locus of the vertex of a right angle whose sides pass through the given points.]

4. Find the equation of a circle which is tangent to the lines $x = 0$, $y = 0$, and $ax + by + c = 0$.

5. A line is drawn through each of the points $(a, 0)$ and $(-a, 0)$, the two lines forming a constant angle θ. Find the equation of the locus of their point of intersection.

6. A straight line moves so that the sum of the perpendiculars drawn to it from two fixed points is constant. Show that it is always tangent to a fixed circle.

7. Give a geometrical construction for the polar of a point with respect to a circle.

8. If the polar of a point P passes through Q, then the polar of Q passes through P.

9. Find the equations of the common tangents of the circles $x^2+y^2=5$ and $x^2 + y^2 - 10x + 20 = 0$.

10. Find the locus of a point which moves so that the length of a tangent drawn from it to one given circle is k times the length of a tangent drawn from it to another given circle.

11. Find the equation of a circle through the points of intersection of $x^2 + y^2 = 4$ and $x^2+y^2-2x+4y+4=0$ and tangent to the line $x-2y=0$.

12. Show that the polars of a given point P with respect to the circles of a pencil pass through a fixed point, unless P is on the line of centers.

13. A point moves so that the sum of the squares of its distances from the sides of a given square is constant. Show that its locus is a circle.

14. A point P moves so that its distance from a fixed point A is always equal to k times its distance from another fixed point B. Show that its locus is a circle, if $k \neq 1$. Show also that for different values of k these circles have a common radical axis.

15. A line rotating about a fixed point O meets a fixed line in a point P. Find the locus of a point Q on OP such that $OP \cdot OQ$ is constant.

16. Prove that among the circles of a pencil there are at most two which are tangent to a given straight line (unless all the circles are tangent to the line). When is there only one? None?

[Hint: Let the given line be the x-axis.]

17. Inversion with Respect to a Circle. Given a circle with center O and radius r. Corresponding to any point P in the plane (distinct from O) there exists a unique point P' on OP such that $OP \cdot OP' = r^2$. The point P' is called the ***inverse*** of P with respect to the given circle. Prove the following propositions:

(*a*) If P' is the inverse of P, P is the inverse of P'.

(*b*) If P is inside the given circle, P' is outside; and vice versa.

(*c*) Every point on the given circle corresponds to itself.

(*d*) If the coördinates of P and its inverse P' are (x, y) and (x', y') respectively, referred to two rectangular axes through O, we have

$$x' = \frac{r^2x}{x^2 + y^2}, \quad y' = \frac{r^2y}{x^2 + y^2}; \quad \text{and} \quad x = \frac{r^2x'}{x'^2 + y'^2}, \quad y = \frac{r^2y'}{x'^2 + y'^2}.$$

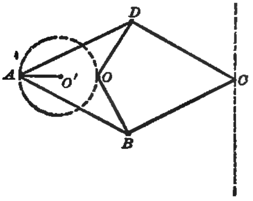
(*e*) If a point P describes a curve, the inverse P' describes a curve called the ***inverse*** of the former curve. The inverse of any straight line through O is this line itself.

(*f*) The inverse of any line not through O is a circle through O, and the inverses of parallel lines are circles tangent at O.

(*g*) The inverse of any circle is a circle, unless the given circle passes through O, in which case its inverse is a straight line.

(*h*) Two orthogonal circles or lines have orthogonal inverses.

(*i*) Any circle orthogonal to the given circle is its own inverse.



(*j*) The adjoining figure illustrates a simple mechanism for changing circular motion into rectilinear motion. It is known as the ***inversor of Peaucellier.*** The heavy lines represent rigid bars, hinged at their extremities. The sides of the quadrilateral $ABCD$ are all equal and $OB = OD = \rho$. Prove that if O is fixed and the mechanism is allowed to move in any way it can, C is always the inverse of A with respect to a circle with center O and radius $r = \sqrt{l^2 - \rho^2}$, where l is the side of the rhombus $ABCD$. Hence, if A describes a circle through O, C will describe a straight line.

CHAPTER XIII

THE CONIC SECTIONS

216. Definition of a Conic. A *conic section** or simply a *conic* is defined as the locus of a point which moves so that its distance from a fixed point, F_1, is always equal to a given constant, e, times its distance from a fixed line D_1D_1'.

The fixed point F_1 is called the ***focus***. The fixed straight line D_1D_1' is called the ***directrix***. The constant e is called the ***eccentricity***. It is assumed that $e > 0$ and that F_1 does not lie on D_1D_1'.

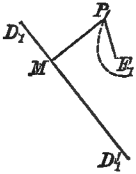


FIG. 189

If P (Fig. 189) is any point on the curve, we have, by the preceding definition,

$$F_1P = e \cdot MP, \tag{1}$$

where MP is the perpendicular distance of P from the directrix. It must be remembered that F_1P and MP are absolute quantities, not directed quantities, and that e is positive.

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* The name "conic section" is due to the fact that the curves in question were originally obtained as the sections of a right circular cone. They were discussed from this point of view by the ancient Greeks.

217. The Equation of a Conic. Let the directrix be chosen as the y-axis and the line through F_1 perpendicular to the directrix as the x-axis (Fig. 190). The coördinates of F_1 may then be taken as $(p, 0)$, where p is different from zero. Let $P(x, y)$ be any point on the conic. Then

$$F_1P = \sqrt{(x-p)^2 + y^2},$$

and

$$MP = +x \text{ or } -x$$

according as x is positive or negative. Equation (1), § 216 then becomes

$$\sqrt{(x-p)^2 + y^2} = \pm ex.$$

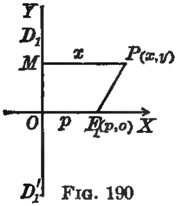


FIG. 190

Squaring both sides of this equation and simplifying, we have

$$(2) \qquad (1-e^2)x^2 + y^2 - 2px + p^2 = 0.$$

This is the equation of the conic. For, the coördinates of every point (x, y) satisfying the definition of the conic will satisfy equation (2), and conversely, every point whose coördinates satisfy equation (2) will satisfy equation (1). Why?

This is an equation of the type considered in § 183. It represents an ***ellipse*** if $1-e^2>0$, a ***hyperbola*** if $1-e^2<0$, and a ***parabola*** if $1-e^2=0$. Hence we have,

A conic is an ***ellipse****, a* ***parabola****, or a* ***hyperbola*** *according as the eccentricity e is* ***less than*** **1**, ***equal to*** **1**, *or* ***greater than*** **1**.

THE ELLIPSE

218. Standard Equation of the Ellipse: $e<1$. We have seen in § 183 how to determine the locus of equation (2) by completing the square. If we apply the same method here, equation (2) may be written in the form

$$(3) \quad \left[x^2 - \frac{2p}{1-e^2}x + \frac{p^2}{(1-e^2)^2}\right] + \frac{y^2}{1-e^2} = -\frac{p^2}{1-e^2} + \frac{p^2}{(1-e^2)^2},$$

or

$$(4)\qquad \left[x-\frac{p}{(1-e^2)}\right]^2+\frac{y^2}{1-e^2}=\frac{p^2e^2}{(1-e^2)^2}.$$

Since $1-e^2$ is positive by hypothesis this equation represents an ellipse whose center is at the point $(p/(1-e^2),\ 0)$, and whose axes coincide with the two straight lines $x=p/(1-e^2)$ and $y=0$ (Fig. 191).

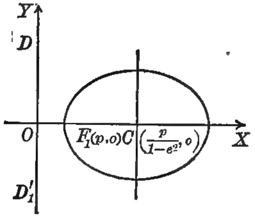


FIG. 191

Let us move the curve parallel to the x-axis through a distance $-p/(1-e^2)$, *i.e.* to the left if $p>0$. Then its center comes to the origin, and its equation becomes

$$(5)\qquad x^2+\frac{y^2}{1-e^2}=\frac{p^2e^2}{(1-e^2)^2},$$

or

$$(6)\qquad \frac{x^2}{\dfrac{p^2e^2}{(1-e^2)^2}}+\frac{y^2}{\dfrac{p^2e^2}{1-e^2}}=1.$$

If we place

$$(7)\qquad \frac{p^2e^2}{(1-e^2)^2}=a^2,\qquad \frac{p^2e^2}{1-e^2}=b^2,$$

the equation of the ellipse in its new position, *i.e.* with its center at the origin (Fig. 192), becomes

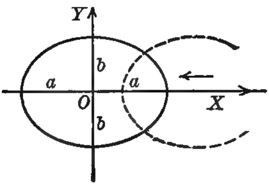


FIG. 192

$$(\mathrm{I}_x)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

From (7) we have

$$(8)\qquad b^2=a^2(1-e^2),$$

which shows that $b<a$, since $e<1$. If the ellipse is given in the form (I_x), a and b are known. Then the value of e can be found in terms of a and b by solving equation (8); this gives

$$e^2 = \frac{a^2 - b^2}{a^2}. \tag{9}$$

219. Properties of the Ellipse. It is important to distinguish between the properties of a curve as such and those properties which are concerned merely with the relations the curve bears to the coördinate axes. Thus the ellipse, as a certain kind of curve, is symmetrical with respect to two perpendicular lines called the ***axes*** of the curve. The longer of the segments on these lines cut off by the curve is called the ***major axis***, the shorter one, the ***minor axis***. The intersection of the two axes of the curve is called the ***center*** of the ellipse.

Every ellipse, no matter how it is situated in the plane of coördinates, has a major axis and a minor axis as well as a center. From the way in which the equation was derived, we know also that every ellipse has a focus and a directrix. The symmetry of the curve with respect to the y-axis shows that this same curve could have been obtained from a second focus F_2 and a second directrix D_2D_2' on the opposite side of the center.

We shall now investigate how the two foci and the two directrices are related to the major axis, the minor axis, and the center.

220. Foci and Directrices. The original position of the focus F_1 was $(p, 0)$; the abscissa of its new position is

$$p - \frac{p}{1-e^2} = -\frac{pe^2}{1-e^2}.$$

Since from (7) we know that $pe/(1-e^2) = a$, we find *the coördinates of the focus F_1 in the new position to be* $(-ae, 0)$.

(See Fig. 193.) Similarly the equation of the directrix D_1D_1' in its new position is

$$x = -\frac{p}{1-e^2},$$

or

$$x = -\frac{a}{e}. \tag{10}$$

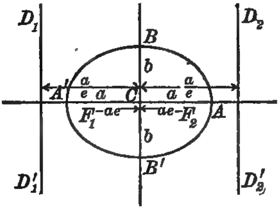


FIG. 193

The second focus F_2 has the coördinates $(ae, 0)$. The second directrix D_2D_2' has the equation

$$x = \frac{a}{e}. \tag{10'}$$

221. The Ellipse in Other Positions. If the center of the ellipse is at the origin and the major axis is on the y-axis, the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \tag{I$_y$}$$

where, as before, $2\,a$ is the length of the major axis and $2\,b$ is the length of the minor axis. The foci of this curve are at the points $(0, ae)$, $(0, -ae)$; the equations of the directrices are $y = \pm\, a/e$.

The equation of an ellipse whose center is at the point (h, k) and whose axes are parallel to the coödinate axes is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \qquad (a > b) \tag{II$_x$}$$

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1, \qquad (a > b) \tag{II$_y$}$$

according as the major axis is parallel to the x-axis or to the y-axis. Finally we can reduce an equation of the form

$$Ax^2 + By^2 + Dx + Ey + C = 0, \quad A > 0,\ B > 0, \tag{III}$$

to the form II$_x$ or II$_y$, if it has a real locus. (See § 183.)

222. The Case $a = b$. The Circle. If $a = b$ the equation (I_x) reduces to the equation of a circle. The relation $a = b$ implies, however, that $e = 0$ and this value of e is excluded in the definition of a conic. On the other hand it is clear that for a given value of a, as the eccentricity approaches zero, the ellipse approaches a circle. At the same time, the foci approach the center, and the directrices recede indefinitely. Why? Since the circle is a limiting form of an ellipse it is classified as an ellipse with equal axes and is counted among the conics.

223. A Geometric Property of an Ellipse. An important geometric property of any ellipse follows from the fact that the distance from the center to either focus, which we shall denote by c, is given by the relation

$$c = ae = \sqrt{a^2 - b^2},$$

or

$$c^2 = a^2 - b^2. \tag{11}$$

This relation shows that c, a, and b are the sides of a right-angled triangle in which a is the hypotenuse (Fig. 194). In other words, *a circle drawn with its center at an extremity of the minor axis and with its radius equal to a, will cut the major axis in the foci, F_1 and F_2.*

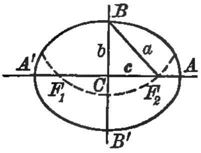


FIG. 194

In computing the elements of an ellipse from a and b, it is generally convenient first to find c from (11) and then to find e from the relation*

$$e = \frac{c}{a}. \tag{12}$$

---

* This relation is equivalent to (9), § 218. It may be expressed by saying that e is the *cosine* of the angle CF_2B, Fig. 194.

The extremities of the major axis are called the ***vertices*** of the ellipse.

The chord through a focus perpendicular to the major axis is called a ***latus rectum.*** Its length is $\mathbf{2\,b^2/a}$. Why?

224. Illustrative Examples. Example 1. Given the ellipse

$$4x^2 + 9y^2 - 36 = 0.$$

Find the coördinates of the center, the vertices, the foci, and the equations of the directrices.

The given equation may be written in the form

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

from which follows that $a = 3$, $b = 2$. Therefore $c = \sqrt{a^2 - b^2} = \sqrt{5}$ and $e = \sqrt{5}/3$. The coördinates of the center are $(0, 0)$, the vertices $(3, 0)$ and $(-3, 0)$, the foci $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$ and the equations of the directrices are $x = -9/\sqrt{5}$ and $x = 9/\sqrt{5}$ (Fig. 195).

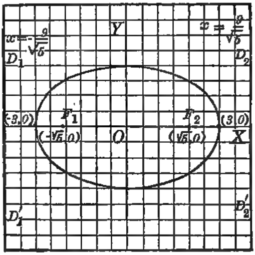


Fig. 195

Example 2. Find the coördinates of the center, the vertices, the foci, and the equations of the directrices of the ellipse

$$25x^2 + 9y^2 - 50x + 36y - 164 = 0.$$

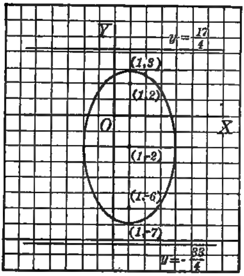


Fig. 196

From § 183, we know that the given equation may be written in the form

$$25(x - 1)^2 + 9(y + 2)^2 = 225,$$

or

$$\frac{(x-1)^2}{9} + \frac{(y+2)^2}{25} = 1.$$

We now conclude that the center is at $(1, -2)$, and that the major axis is parallel to the y-axis. Here $a = 5$, $b = 3$, $c = 4$, $e = \frac{4}{5}$ and $a/e = \frac{25}{4}$. Sketching the ellipse we find from the figure that the vertices are $(1, 3)$ and $(1, -7)$, and the foci $(1, 2)$ and $(1, -6)$. The equations of the directrices are $y = 17/4$, $y = -33/4$.

EXERCISES

In the following ellipses determine the major axis, the minor axis, the coördinates of the center, the coördinates of the vertices and foci, and the equations of the directrices. Sketch the curves.

1. $3x^2 + 4y^2 = 12$.
2. $4x^2 + 3y^2 = 12$.
3. $4x^2 + y^2 = 16$.
4. $36x^2 + 25y^2 = 144$.
5. $2x^2 + 4y^2 = 3$.
6. $5x^2 + y^2 = 75$.
7. $3x^2 + 3y^2 = 12$.
8. $x^2 + 2y^2 = 8$.
9. $4x^2 + 9y^2 - 16x - 18y - 23 = 0$.
10. $9x^2 + 25y^2 - 150y = 0$.
11. $4x^2 + y^2 - 8x + 4y + 4 = 0$.
12. $9x^2 + 4y^2 + 36x - 16y + 16 = 0$.

13. Write the equation of the following ellipses:

(*a*) Center at origin, major axis $= 4$ on x-axis, minor axis $= 3$.

(*b*) Center at origin, major axis $= 5$ on y-axis, minor axis $= 3$.

(*c*) Center at origin, major axis $= 6$, minor axis $= 3$ (two solutions).

(*d*) Center at origin, eccentricity $4/5$, foci at $(-2, 0)$ and $(2, 0)$.

(*e*) Center at $(1, 2)$, major axis $= 6$ parallel to x-axis, minor axis $= 4$.

(*f*) Foci at $(0, 2)$ and $(0, 8)$, major axis $= 10$.

14. An ellipse has its center at the origin, and its axes coincide with the coördinate axes. The ellipse passes through the points $(\sqrt{7}, 0)$ and $(2, 1)$. Find its equation.

[Hint. Assume the equation of the ellipse in the form (I_x). Find a and b from the fact that the ellipse must pass through the given points.]

15. Find the equation of the ellipse symmetrical with respect to the coördinate axes if the major axis is twice the minor axis and the curve passes through the point $(2, 1)$. How many solutions?

16. Show that the equation of the ellipse whose vertex is at the origin and whose major axis is on the x-axis is of the form $a^2y^2 = b^2(2ax - x^2)$.

17. Verify equation (I_x) by deriving the equation of a conic whose focus is at $(-ae, 0)$ and whose directrix is the line $x = -a/e$.

18. Find the equation of the ellipse whose focus is at $(0, 0)$, whose directrix is the line $x + y - 1 = 0$ and whose eccentricity is $1/2$.

19. Find the equation of the ellipse whose eccentricity is $1/3$, whose focus is at $(3, 1)$ and whose directrix is the line $3x + 4y - 1 = 0$.

20. Find the equation of the conic whose focus is at $(2, 1)$, whose eccentricity is 3, and whose directrix is the line $3x + y = 1$. What kind of a conic is the curve?

225. Focal Radii. The segments F_1P and F_2P joining any point P on an ellipse to the foci F_1, F_2, are called the ***focal radii*** of the point P.

If the equation of the ellipse is given in the standard form (I_x), *the focal radii of any point* $P(x_1, y_1)$ *are* $a - ex_1$, $a + ex_1$.

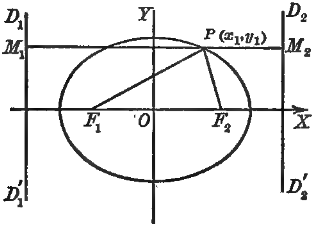


FIG. 197

For, from the definition of an ellipse (Fig. 197),

$$F_1P = e \cdot M_1P, \quad F_2P = e \cdot PM_2$$

But from the figure, we have also

$$M_1P = \frac{a}{e} + x_1, \quad PM_2 = \frac{a}{e} - x_1.$$

Therefore the focal radii are

$$F_1P = a + ex_1, \quad F_2P = a - ex_1.$$

From these relations follows the important property:

The sum of the focal radii of any point of an ellipse is constant and is equal to the major axis $2a$.

It may be noted that this relation still holds when the ellipse is a circle ($e = 0$), although the method of its derivation is not applicable in this case. An ellipse could, therefore, be defined as *the locus of a point which moves so that the sum of its distances from two fixed points (the foci) is constant.*

226. Geometric Constructions of the Ellipse. The property of the ellipse derived in § 225 gives the construction indicated in Fig. 198 for the points of the ellipse when the foci and the major axis are given.

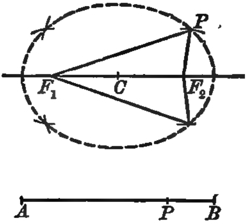


FIG. 198

The segment AB is the major axis. Different positions of P on this segment give corresponding values AP and PB of the focal radii of a point on the ellipse. Circles drawn with these radii and centers at the foci intersect in points of the ellipse. To each position of P on AB correspond four points of the ellipse.

A very convenient method of drawing an ellipse is indicated in Fig. 199. Two pins are stuck in the paper at the foci and

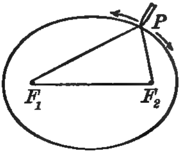


FIG. 199

a loop of thread thrown over them. If a pencil point is inserted in the loop and moved so as to keep the thread taut, it will describe an ellipse. Why?

Another method of constructing an ellipse (much used by draftsmen) is based on the fact (§ 179) that if the ordinates of the circle $x^2 + y^2 = a^2$ are shortened in the ratio $b : a$ $(b < a)$

there results an ellipse with major axis $2a$ and minor axis $2b$. The adjoining figure (Fig. 200) exhibits the method. Explain and prove the method correct.

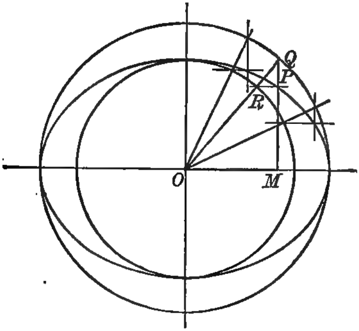


FIG. 200

[HINT. The two circles being of radii b and a respectively, we have $OR/OQ = b/a$; hence, $MP/MQ = b/a$. Why ?]

EXERCISES

1. Construct an ellipse whose foci are 2 inches apart and whose major axis measures 3 inches.

2. Construct an ellipse whose major and minor axes are 2 and 1.5 inches respectively.

3. From the property of § 225 derive the equation of an ellipse.

4. From Fig. 200 show that the coördinates (x, y) of any point on the ellipse (I_x), p. 339, are given by the equations

$$x = a \cos \theta, \quad y = b \sin \theta,$$

where θ is the angle MOQ. Do these values of x, y satisfy the equation of the ellipse for all values of θ?

5. From the relation between the ordinates of a circle and an ellipse whose major axis is equal to the diameter of the circle prove that any plane section of a circular cylinder is an ellipse, provided the plane of section is not parallel to an element of the cylinder.

6. Prove from the result of the last exercise that a properly determined plane section of an elliptic cylinder is a circle.

THE HYPERBOLA

227. Standard Equations of the Hyperbola. If $e > 1$, then $1 - e^2 < 0$, and it is convenient to write (2), § 217, in the form

$$(e^2 - 1)x^2 - y^2 + 2\,px - p^2 = 0. \tag{13}$$

Completing the square and transforming as in § 218, we obtain

$$\left(x + \frac{p}{e^2 - 1}\right)^2 - \frac{y^2}{e^2 - 1} = \frac{p^2e^2}{(e^2 - 1)^2}.$$

This equation represents a hyperbola whose center is at the point $(-p/(e^2 - 1),\ 0)$ and whose axes coincide with the lines $x = -p/(e^2 - 1)$, and $y = 0$ (Fig. 201).

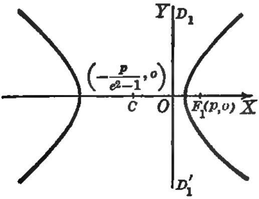


FIG. 201

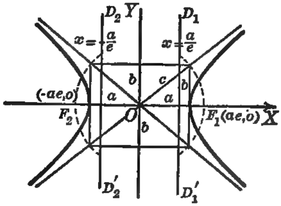


FIG. 202

If the curve is moved parallel to the x-axis so that its center coincides with the origin (Fig. 202), its equation becomes

$$\frac{x^2}{\dfrac{p^2e^2}{(e^2 - 1)^2}} - \frac{y^2}{\dfrac{p^2e^2}{e^2 - 1}} = 1.$$

If, then, we place

$$\frac{p^2e^2}{(e^2 - 1)^2} = a^2, \quad \frac{p^2e^2}{e^2 - 1} = b^2, \tag{14}$$

the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{I$_x$}$$

From (14) we have the relation connecting a, b, e as

$$b^2 = a^2(e^2 - 1),$$

or

$$e^2 = \frac{a^2 + b^2}{a^2}. \tag{15}$$

Here, as in the case of the ellipse, it is important to note some of the properties of the curve. It is seen that the locus is symmetrical with respect to the line passing through the focus and perpendicular to the directrix. This line is called the ***principal axis*** and the segment of this line intercepted by the curve is called the ***transverse axis*** and its length is $2a$. The extremities of the transverse axis are called the ***vertices***, and the point midway between the vertices is called the ***center***. The curve is also symmetrical with respect to the line through the center and perpendicular to the transverse axis. The segment on this line whose length is $2b$ and whose mid-point is at the center of the hyperbola is called the ***conjugate axis***.

If a hyperbola has its center at the origin, and if its transverse axis $2a$ is on the y-axis, and its conjugate axis is $2b$, its equation is

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1. \tag{I$_y$}$$

The equation of a hyperbola whose center is at the point (h, k), whose transverse axis is $2a$, and whose conjugate axis is $2b$, is

$$(\text{II}_x)\quad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \quad \text{or} \quad (\text{II}_y)\quad \frac{(x-h)^2}{b^2} - \frac{(y-k)^2}{a^2} = -1,$$

according as the transverse axis is parallel to the x-axis or the y-axis.

The equation of any hyperbola with axes parallel to the coördinate axes may be written in the form

$$(\text{III})\qquad Ax^2 + By^2 + Dx + Ey + C = 0, \quad A > 0, \ B < 0;$$

and every equation of this form ($A > 0$, $B < 0$) represents a hyperbola or a pair of straight lines (cf. § 183).

As in the case of the ellipse, it is easy to show that every hyperbola has two foci on the transverse axis, one on each side of the center and at a distance c from the center, where

$$c^2 = a^2e^2 = a^2 + b^2.$$

With each focus is associated a directrix perpendicular to the transverse axis and at a distance a/e from the center (Fig. 202).

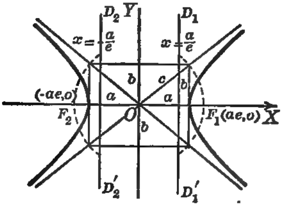


FIG. 202 (repeated)

The ***latus rectum***, *i.e.* the chord through the focus and perpendicular to the transverse axis prolonged, is of length $2b^2/a$. The ***asymptotes*** of the hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

are the lines

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0. \tag{16}$$

228. Geometric Properties of the Hyperbola. *The segment from the center to a focus of a hyperbola is the hypotenuse of a right-angled triangle whose legs are the semi-transverse and semi-conjugate axes.* Why? It is readily seen, moreover, that, if a rectangle be constructed by drawing lines through the extremities of each axis parallel to the other axis, the diagonals (extended) of this rectangle are the *asymptotes* of the hyperbola (Fig. 202). The circle drawn on either diagonal as a diameter passes through the foci. Why?

229. Illustrative Examples.

EXAMPLE 1. Find the coördinates of the center, the vertices, and the foci, and the equations of the directrices and the asymptotes of the hyperbola

$$4x^2 - 9y^2 + 36 = 0.$$

The equation is readily transformed into the form

$$\frac{x^2}{9} - \frac{y^2}{4} = -1.$$

It is now seen that the center is at the origin and that the transverse axis is along the y-axis (Fig. 203). The vertices are $(0, 2)$ and $(0, -2)$. Since $c = \sqrt{13}$, the coördinates of the foci are $(0, \sqrt{13})$ and $(0, -\sqrt{13})$. The asymptotes are given by $4x^2 - 9y^2 = 0$ or $2x - 3y = 0$ and $2x + 3y = 0$. Since $e = \sqrt{13}/2$ the equations of the directrices are

$$y = \pm \frac{4}{\sqrt{13}} = \pm \frac{4}{13}\sqrt{13}.$$

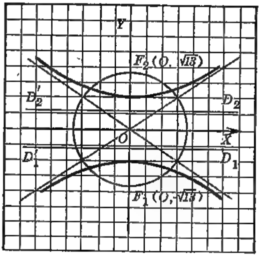


FIG. 203

EXAMPLE 2. Find the coördinates of the center, the foci, and the vertices, and the equations of the asymptotes and the directrices of the hyperbola

$$16x^2 - 9y^2 + 32x + 54y - 209 = 0.$$

The given equation may be written in the form

$$16(x+1)^2 - 9(y-3)^2 = 144,$$

or

$$\frac{(x+1)^2}{9} - \frac{(y-3)^2}{16} = 1.$$

The center is therefore at the point $(-1, 3)$ and the transverse axis is parallel to the x-axis (Fig. 204). Since $a = 3$, the vertices are $(2, 3)$ and $(-4, 3)$. Moreover, since $c = \sqrt{9 + 16} = 5$, the foci are at the points $(4, 3)$ and $(-6, 3)$. Likewise, $e = c/a = 5/3$ and hence the directrices are $x = -\frac{14}{5}$, $x = \frac{4}{5}$. The asymptotes are given by

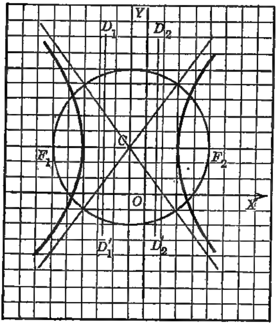


FIG. 204

$$16(x+1)^2 - 9(y-3)^2 = 0.$$

Why? That is, the asymptotes are the lines

$$4x - 3y + 13 = 0,$$

and

$$4x + 3y - 5 = 0.$$

EXERCISES

For each of the following hyperbolas determine the transverse axis, the conjugate axis, the coördinates of the center, the coördinates of the vertices and the foci, and the equations of the directrices and asymptotes. Sketch the curves.

1. $3x^2 - 4y^2 = 12.$

2. $4x^2 - 3y^2 = 12.$

3. $4x^2 - 3y^2 = -12.$

4. $3x^2 - 4y^2 = -12.$

5. $-36x^2 + 25y^2 = 144.$

6. $x^2 - y^2 = 1.$

7. $-9x^2 + y^2 = 36.$

8. $y^2 - 2x^2 = 4.$

9. $4x^2 - 12y^2 - 8x - 24y - 56 = 0.$

10. $5x^2 - 4y^2 + 10x + 25 = 0.$

11. $9x^2 - 16y^2 + 18x - 96y - 279 = 0.$

12. $x^2 - y^2 + 2x - 2y = 2.$

13. Write the equations of the following hyperbolas:

(*a*) Center at origin, transverse axis $= 6$ on x-axis, conjugate axis $= 4$.

(*b*) Center at origin, transverse axis $= 8$ on y-axis, conjugate axis $= 10$.

(*c*) Center at origin, transverse axis and conjugate axis $= 4$, axes coinciding with coördinate axes. Two solutions.

(*d*) Center at origin, focus at $(5, 0)$ and transverse axis $= 8$.

(*e*) Center at origin, transverse axis $= 8$, focus at $(0, 5)$.

(*f*) Center at origin, focus at $(5, 0)$, conjugate axis $= 8$.

(*g*) Center at $(1, 2)$, transverse axis $= 6$ parallel to x-axis, conjugate axis $= 4$.

(*h*) Center at $(0, 3)$, focus at $(0, 5)$, conjugate axis $= 2\sqrt{3}$.

(*i*) Foci at $(1, 2)$ and $(1, -8)$, transverse axis $= 6$.

14. A hyperbola has its center at the origin and its axes on the coördinate axes; it passes through the points $(0, \sqrt{3})$ and $(2, 3)$. Find its equation.

[Hint. Since one point of the hyperbola lies on the y-axis, the equation may be assumed in the form I_y, *i.e.*

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$$

and a and b may then be determined.]

15. Show that the equation of any hyperbola whose vertex is at the origin and whose transverse axis is on the x-axis is of the form $a^2y^2 = b^2(2ax + x^2)$. (See Ex. 16, p. 344.)

16. A hyperbola whose asymptotes are at right angles is called **rectangular**. Prove that the equation of a rectangular hyperbola may be written in the form $x^2 - y^2 = a^2$.

230. Focal Radii of the Hyperbola. If $P(x_1, y_1)$ is any point on the hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the focal radii F_1P and F_2P are given by the equations

$$F_1P = ex_1 + a, \quad F_2P = ex_1 - a.$$

The proof of the above statement is left as an exercise. It is analogous to the corresponding proof in the case of the ellipse (§ 225).

Hence, *the difference of the focal radii of any point on a hyperbola is a constant.*

A hyperbola could, therefore, be defined as the locus of a point which moves so that the difference of its distances from two fixed points (the foci) remains constant.

231. Conjugate Hyperbolas. Any hyperbola determines uniquely a second hyperbola whose transverse and conjugate axes coincide in position and length with the conjugate and transverse axes respectively of the first hyperbola (Fig. 205). Thus, if the equation of the first hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the equation of the second hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

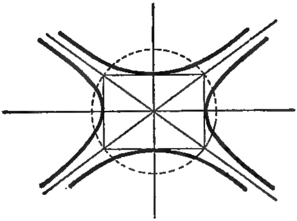


FIG. 205

Each of the two hyperbolas thus related is called the ***conjugate*** of the other, and the two hyperbolas are called ***conjugate hyperbolas.***

Two conjugate hyperbolas have the same asymptotes. Why?

EXERCISES

1. *Geometric construction of the hyperbola.* Show how to construct a hyperbola given the foci and the length of the transverse axis by a method depending on the property of the hyperbola derived in § 230 and entirely analogous to the first method described in § 226 for constructing the ellipse.

2. Derive the equation of the hyperbola from the definition suggested at the end of § 230. [Let the foci be $F_1(c, 0)$ and $F_2(-c, 0)$ and let the constant difference of F_1P and F_2P be $2a$.]

3. What is the equation of the hyperbola $x^2 - y^2 = a^2$ after it has been rotated about the origin through an angle of 45° ? (Cf. § 190.)

4. From the result of Ex. 3 determine the length of the transverse axis of the hyperbola $xy = k$.

5. What are the equations of the hyperbolas conjugate to the hyperbolas in Exs. 1–12, p. 352 ?

6. Prove that the foci of two conjugate hyperbolas are on a circle.

THE PARABOLA

232. Standard Equations of the Parabola.

If in § 217 we let $e = 1$, equation (2) becomes

$$y^2 - 2px + p^2 = 0. \tag{17}$$

or

$$y^2 = 2p\left(x - \frac{p}{2}\right). \tag{18}$$

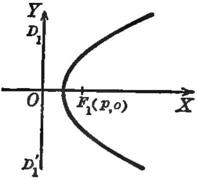


FIG. 206

We saw in § 183 that this equation represents a parabola whose vertex is at the point $(p/2, 0)$ and whose axis coincides with the line $y = 0$ (Fig. 206). If the curve is moved parallel to the x-axis so that its vertex coincides with the origin, the equation of the curve becomes

$$y^2 = 2px. \tag{I$_x$}$$

The focus of the curve is now at the point $(p/2, 0)$ and its directrix is the line $x = -p/2$ (Fig. 207).

The following theorems follow directly. Their proofs are left as exercises.

The equation of a parabola whose vertex is at the origin and whose axis coincides with the y-axis is

$$x^2 = 2py. \tag{I$_y$}$$

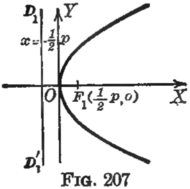


FIG. 207

The equation of a parabola whose vertex is at the point (h, k) and whose axis is parallel to the x-axis is

$$(y - k)^2 = 2p(x - h). \tag{II$_x$}$$

The equation of the parabola whose vertex is at the point (h, k) and whose axis is parallel to the y-axis, is

$$(x - h)^2 = 2p(y - k). \tag{II$_y$}$$

The equation of any parabola whose axis is parallel to the x-axis is of the form

$$By^2 + Dx + Ey + C = 0. \tag{III$_x$}$$

The equation of any parabola whose axis is parallel to the y-axis is of the form

$$Ax^2 + Dx + Ey + C = 0. \tag{III$_y$}$$

The distance from the vertex to the focus and from the directrix to the vertex of the parabola $y^2 = 2px$ is $p/2$.

233. Geometric Properties of the Parabola. The chord drawn through the focus and perpendicular to the axis is called the ***latus rectum.*** *Its length is twice the distance from the focus to the directrix.*

The focal radius connecting any point $P(x_1, y_1)$ on the parabola $y^2 = 2px$ to the focus is equal to $x_1 + p/2$.

The proofs of these properties are left as exercises.

234. Illustrative Examples. EXAMPLE 1. Given the parabola $x^2 = 6y$. Find the coördinates of the vertex and the focus, and the equation of the directrix. Sketch the curve.

The vertex is at (0, 0) and the axis of the curve coincides with the y-axis (Fig. 208). The distance from vertex to focus is 3/2. Therefore the focus is at (0, 3/2). Likewise, the distance from vertex to directrix is 3/2. Hence the equation of the directrix is $y = -3/2$. To sketch the curve, mark the focus, draw the latus rectum and then sketch the curve.

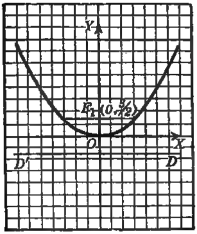


FIG. 208

EXAMPLE 2. Given the parabola $y^2 = -8x + 2y + 15$. Find the coördinates of the vertex and the focus, and the equation of the directrix. Sketch the curve.

The given equation may be written as

$$(y - 1)^2 = -8(x - 2).$$

Therefore the vertex is at (2, 1) (Fig. 209), and the axis is parallel to the x-axis. The distance from vertex to focus and from directrix to vertex is -2. Therefore the focus is at (0, 1) and the equation of the directrix is $x = 4$. The curve is readily sketched by plotting the focus and marking off the latus rectum. It may also be sketched by plotting another point or two.

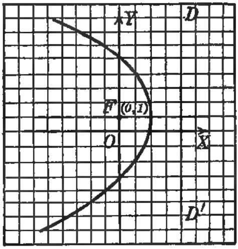


FIG. 209

EXERCISES

Sketch each of the following parabolas. Determine the coördinates of the vertex and the focus, and the equation of the directrix.

1. $y^2 = 4x$.

2. $y^2 = -4x$.

3. $y^2 = 4x + 2$.

4. $y^2 = -4x + 2$.

5. $x^2 = 4y$.

6. $x^2 = -4y$.

7. $x^2 = 4y + 2$.

8. $x^2 = -4y + 2$.

9. $y^2 = 6x + 12$.

10. $x^2 + 4x - 4y + 6 = 0$.

11. $y^2 - 2x - 4y - 8 = 0$.

12. $x^2 + y + 1 = 0$.

13. $y^2 = -4x + 2y + 8$.

14. $y^2 + 2x - 4y = 0$.

15. $x^2 - 2x + 2y = 0$.

16. Write the equation of each of the following parabolas:

(*a*) Vertex at $(0, 0)$ and focus at $(2, 0)$.

(*b*) Vertex at $(0, 0)$, axis coinciding with y-axis, curve passing through the point $(8, 4)$.

(*c*) Focus at $(-1, 3)$ and directrix the line $x - 1 = 0$.

(*d*) Vertex at $(1, -2)$, axis parallel to x-axis, distance from vertex to focus equal to 2.

(*e*) Vertex at $(0, 2)$, directrix parallel to x-axis and parabola passing through the point $(2, 1)$.

235. The Intersections of Conics and Straight Lines. The coördinates of the points of intersection of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2 \tag{19}$$

and the straight line

$$y = mx + k, \tag{20}$$

are found by solving these two equations simultaneously for (x, y). Eliminating y, we obtain the quadratic equation

$$(b^2 + a^2m^2)x^2 + 2\,a^2mkx + a^2(k^2 - b^2) = 0, \tag{21}$$

the roots of which are the abscissas of the points of intersection. For each of these roots the corresponding ordinate is found by substituting in (20). Why not in (19)? We accordingly obtain, in general, two solutions (x, y). These solutions are real and distinct, real and equal, or imaginary, according as

$$b^2 + a^2m^2 - k^2 > 0,\ = 0,\ \text{or} < 0. \tag{22}$$

Corresponding to these three cases, the straight line intersects the ellipse in two distinct points, in two coincident points (*i.e.* in a single point), or not at all.

The discussion just given includes for $a = b$ the case of the intersection of a circle and a straight line.

To treat the intersection of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ with the straight line (20), we need only notice that algebraically we can reduce this problem to the preceding by simply writing $-b^2$ for b^2. Why?

This leads to the equation

$$(a^2m^2 - b^2)x^2 + 2\,a^2mkx + a^2(k^2 + b^2) = 0.$$

This is a quadratic equation unless $a^2m^2 - b^2 = 0$. If $a^2m^2 - b^2 = 0$, the line (20) is parallel to an asymptote, and, if $k \neq 0$, it meets the hyperbola in only one point. If $k = 0$ the line is an asymptote and does not meet the curve at all. If $a^2m^2 - b^2 \neq 0$, we conclude that the line (20) intersects the hyperbola in two distinct points, two coincident points (*i.e.* in only one point), or not at all, according as

$$k^2 - a^2m^2 + b^2 > 0, \ = 0, \ < 0. \tag{23}$$

Finally, the line (20) will meet the parabola

$$y^2 = 2\,px, \tag{24}$$

in the points whose abscissas are the roots of the equation

$$m^2x^2 + 2(mk - p)x + k^2 = 0.$$

If $m = 0$, the line meets the curve in only one point. If $m \neq 0$, the line will intersect the parabola in two distinct points, two coincident points, or not at all, according as

$$p - 2\,mk > 0, \ = 0, \text{ or } < 0. \tag{25}$$

Similar results are evidently secured also for straight lines $x = k$, parallel to the y-axis. We then have the theorem:

Any conic is met by a straight line in the plane of the conic in two distinct points, a single point, or not at all.

EXERCISES

1. Draw figures illustrating all the results of the last article.

2. In a manner similar to that of the last article discuss the intersections of the line $y = mx + k$ and the conic $y^2 = 2\,px - gx^2$.

3. Derive conditions analogous to (22), (23), and (25) of the last article when the straight line is assumed in the form $Ax + By + C = 0$. These conditions are slightly more general than those given in the text. Why?

236. Tangents and Normals. Slope Forms. If, for a given value of m, the value of k in the equation $y = mx + k$ is so determined that the intersections of the line $y = mx + k$ with a given conic coincide, *i.e.* so that the quadratic equation determining the abscissas of the points of intersection has equal roots, the line will be tangent to the conic. Why? (See § 209.) The slope forms of the equations of the tangents to a conic result directly from the middle one of each of the conditions (22), (23), and (25) for the determination of k. Hence the equation of the tangent whose slope is m is:

for the ellipse $b^2x^2 + a^2y^2 = a^2b^2$,

$$y = mx \pm \sqrt{a^2m^2 + b^2}; \tag{26}$$

for the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$,

$$y = mx \pm \sqrt{a^2m^2 - b^2}; \tag{27}$$

for the parabola $y^2 = 2\,px$,

$$y = mx + \frac{p}{2\,m}. \tag{28}$$

We note that for a given slope the parabola has only one tangent, the ellipse two, and the hyperbola either two or none according as $a^2m^2 - b^2 > 0$ or < 0. [The condition $a^2m^2 - b^2 = 0$ yields the asymptotes.]

The line drawn perpendicular to a tangent through its point of contact P is called the ***normal*** at P.

EXERCISES

1. Find the equations of the tangents to the following conics satisfying the conditions given, and find for each tangent its point of contact:

(*a*) $4\,x^2 + 9\,y^2 = 36$, $m = \frac{1}{2}$.
(*b*) $y^2 = 3\,x$, inclination 30°, 45°, 135°.
(*c*) $9\,x^2 - 25\,y^2 = 225$, perpendicular to $x + y + 1 = 0$.
(*d*) $x^2 - y^2 = 1$, parallel to $5\,x + 3\,y - 10 = 0$.
(*e*) $y^2 = 8\,x$, perpendicular to $2\,x - 3\,y + 6 = 0$.

2. Show that the line $y = mx \pm \sqrt{b^2 - a^2m^2}$ is tangent to the hyperbola $b^2x^2 - a^2y^2 + a^2b^2 = 0$ for all real values of m for which $b^2 - a^2m^2 > 0$.

3. For what value of k will the line $y = 2x + k$ be tangent to the hyperbola $x^2 - 4y^2 - 4 = 0$?

4. Find the coördinates of the points of intersection of the line $3x - y + 1 = 0$ and the ellipse $x^2 + 4y^2 = 65$.

5. Find the points of contact of the tangents $y = mx \pm \sqrt{a^2m^2 + b^2}$ to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

6. From the result of Ex. 5 find the equations of the normals to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ whose slope is m.

7. By the method suggested in Exs. 5 and 6, find the equation of the normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ in terms of its slope.

8. Same problem as Ex. 7 for the parabola $y^2 = 2px$.

9. A tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ will pass through the point (x_1, y_1), if $y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$. By solving this equation for m show that through a given point (x_1, y_1) will pass two distinct tangents, one tangent, or no tangents, according as $b^2x_1^2 + a^2y_1^2 - a^2b^2 > 0$, $= 0$, or < 0.

10. By the method of Ex. 9, discuss the number of tangents that can be drawn from a given point (x_1, y_1) to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$; to the parabola $y^2 = 2px$.

11. Find the equations of the tangents to the parabola $y^2 = 4x$ which pass through the point $(-2, -2)$.

12. Find the equations of the tangents to the ellipse $4x^2 + y^2 = 16$ which pass through the points $(\sqrt{3}, 2)$; $(0, 4)$; $(0, 8)$.

237. Tangents. Point Form. The slope of the curve

$$Ax^2 + By^2 + Dx + Ey + C = 0, \tag{29}$$

at a point (x_1, y_1) *on the curve*, was found in § 184 to be

$$m = -\frac{2Ax_1 + D}{2By_1 + E}.$$

Hence the equation of the tangent to (29) at (x_1, y_1) is

$$y - y_1 = -\frac{2Ax_1 + D}{2By_1 + E}(x - x_1).$$

This reduces to

$$2Ax_1x + 2By_1y + Dx + Ey = 2Ax_1^2 + 2By_1^2 + Dx_1 + Ey_1 \tag{30}$$

Since (x_1, y_1) is by hypothesis on the curve (29), we have

$$2\,Ax_1^2 + 2\,By_1^2 = -2\,Dx_1 - 2\,Ey_1 - 2\,C.$$

Substituting this value in the right-hand member of (30),

$$2\,Axx_1 + 2\,By_1y + Dx + Ey = -Dx_1 - Ey_1 - 2\,C.$$

Hence, by transposing and dividing by 2, we obtain the equation of the tangent to (29) at the point (x_1, y_1) in the form

$$Ax_1x + By_1y + D\frac{(x + x_1)}{2} + E\frac{(y + y_1)}{2} + C = 0. \tag{31}$$

This equation is readily written down from (29) by replacing x^2, y^2, x, and y by x_1x, y_1y, $\frac{1}{2}(x + x_1)$, and $\frac{1}{2}(y + y_1)$, respectively. By applying this rule to the standard equations of the conics which are special cases of (29) we obtain:

The equation of the tangent at the point (x_1, y_1)

to the ellipse $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad$ is $\quad \dfrac{x_1x}{a^2} + \dfrac{y_1y}{b^2} = 1;$

to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad \frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1;$$

to the parabola

$$y^2 = 2px \quad \text{is} \quad y_1y = p(x + x_1).$$

EXERCISES

1. Write the equation of the tangent to each of the following conics at the point indicated:

(*a*) $x^2 + 4\,y^2 = 8$, at $(2, 1)$.
(*b*) $4\,x^2 - 3\,y^2 = 9$, at $(3, -3)$.
(*c*) $y^2 - 6\,x = 0$, at the point where $y = -3$.
(*d*) $x^2 - y^2 = 4$, at $(2, 0)$.
(*e*) $x^2 - 2\,y^2 = -4$, at $(-2, 2)$.
(*f*) $y^2 - 4\,x = 0$, at the extremities of the latus rectum.

2. Write the equation of the normal to each of the conics in Ex. 1 at the point indicated.

3. Find the equation of the normal to each of the conics $b^2x^2 + a^2y^2 = a^2b^2$, $b^2x^2 - a^2y^2 = a^2b^2$, and $y^2 = 2\,px$ at the point (x_1, y_1).

4. Prove that the tangents drawn to an ellipse at the extremities of any diameter (chord through the center) are parallel.

5. Show that an ellipse and a hyperbola with common foci intersect at right angles.

6. Show that the tangents at the vertices of a hyperbola meet the asymptotes in points at the same distance from the center as are the foci.

7. Find the angle (in degrees and minutes) at which the two curves $x^2 + 2\,y^2 = 9$ and $y^2 + 4\,x = 0$ intersect.

8. Show that the secant of the parabola $y^2 = 2\,px$ joining the points (x_1, y_1) and (x_2, y_2) on the curve has the equation $2\,px - (y_1 + y_2)y + y_1y_2 = 0$. Show that this reduces to the equation of the tangent when the given points coincide.

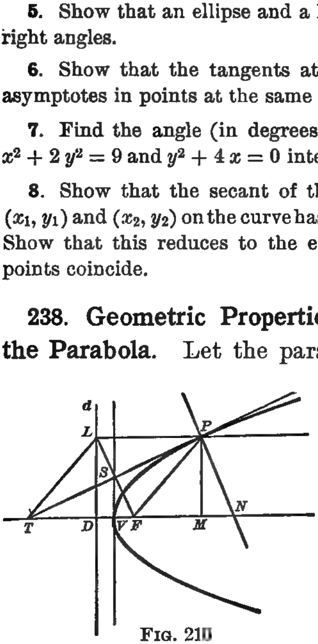


FIG. 210

238. Geometric Properties of Tangents and Normals to the Parabola. Let the parabola have the focus F, the vertex V, and the directrix d, the latter meeting the axis VF in D (Fig. 210). If the vertex is chosen as origin of a system of rectangular coördinates and the axis is chosen as the x-axis, while the segment DF is denoted by p, the equation of the parabola is $y^2 = 2\,px$. Now let $P(x_1, y_1)$ be any point on the parabola. The equation of the tangent at this point is $y_1y = p(x + x_1)$. This tangent meets the axis of the parabola (the x-axis) in the point $T(-x_1, 0)$. Hence

$$TV = VM,$$

where M is the foot of the perpendicular dropped from P on the axis. From this, and by the definition of the parabola, follow the relations

$$TF = DM = LP = FP,$$

where L is the foot of the perpendicular drawn from P to the directrix. Hence *$TFPL$ is a rhombus.* We conclude further that

$$\angle LPT = \angle TPF;$$

and, if S is the intersection of the diagonals of the rhombus $TFPL$, that the angle *FSP is a right angle.* Moreover, the line drawn through V, the mid-point of TM, perpendicular to TM, passes through S. We have then the following theorems:

Theorem 1. *The tangent to a parabola at any point P bisects one of the angles formed by the focal radius of P and the line through P parallel to the axis of the parabola; the normal at P accordingly bisects the other angle.*

Theorem 2. *The foot of the perpendicular dropped from the focus on any tangent to the parabola is on the tangent at the vertex.*

EXERCISES

1. Prove theorems 1 and 2 of § 238 analytically.

2. Give a geometric construction for the tangent to a given parabola at a given point. (The axis of the curve as well as the curve is supposed to be given.)

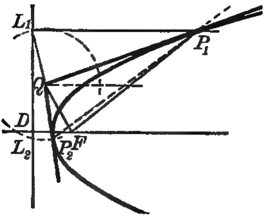
[A geometric construction means a construction with ruler and compass.]

3. Given the focus and directrix of a parabola, show how any number of points of the parabola can be constructed on the basis of the results of the last article.

4. Given the focus of a parabola and the tangent at the vertex, use Theorem 2 of § 238 to draw any number of tangents to the parabola. These tangents will give a vivid picture of the shape of the curve; the tangents are said to ***envelop*** the curve. The curve itself is not supposed to be given.

5. The outline and axis of a parabola are given; show how to construct the focus and directrix.

6. To construct the tangents to a given parabola from a given external point. Assume that the focus and directrix and hence the axis are given.

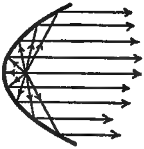


[ANALYSIS: If Q is the given point, it follows from Theorem 1 of the last article that $\Delta\, QL_1P_1$ and $\Delta\, QFP_1$ are congruent. Hence,

$$QL_1 = QF.$$

We determine L_1 (and L_2), therefore, as the intersection with the directrix of the circle with center Q and radius QF. Complete the construction. How is the construction affected when Q assumes various positions in the plane? When is the construction impossible and why? What happens when Q is on the curve?

7. In the figure of Ex. 6, prove that the line through Q parallel to the axis bisects the "chord of contact" P_1P_2.



8. If a parabola is rotated about its axis the surface generated is called a ***paraboloid of revolution.*** Prove that if a source of light is placed at the focus of such a paraboloid*, all the rays issuing from the source will be reflected in the same direction (parallel to the axis of the paraboloid). This is the principle of the so-called parabolic reflectors, used on searchlights, etc.

9. By an argument similar to that employed in § 212, prove that the equation of the chord of contact of the tangents drawn from an external point (x_1, y_1) to the parabola $y^2 = 2px$ is $y_1y = p(x + x_1)$. This line is called the ***polar*** of the given point with respect to the parabola. It is defined by its equation even when no tangents can be drawn through the given point.

10. Prove that the polar of a point Q is parallel to the tangent at the point in which the line through Q parallel to the axis meets the parabola.

11. Prove that the length of the so-called ***subnormal*** MN of a parabola at the point P (see Fig. 210) is independent of the position of P on the curve.

12. Prove (Fig. 210) that $TF = FN = FP$ and that $FS = \frac{1}{2} PN$.

---

* The focus of the generating parabola is called the focus of the paraboloid.

13. Use the relation $FN = FP$ (Ex. 12) to show how to construct the normal at a given point P of a parabola (the focus and axis also being given). Construct a considerable number of normals in this way and show that they envelop a curve. (See Ex. 4 for the meaning of "envelop.")

14. Show that any two perpendicular tangents to a parabola intersect on the directrix.

239. Geometric Properties of Tangents and Normals to the Ellipse.

If for *any* ellipse we let the coördinate axes coincide with the axes of the curve, the equation of the ellipse has the form

$$b^2x^2 + a^2y^2 = a^2b^2.*$$

The equation of the tangent at any point $P_1(x_1, y_1)$ is

$$b^2x_1x + a^2y_1y = a^2b^2.$$

The x-intercept (Fig. 211) of this tangent is

$$OT = \frac{a^2}{x_1}.$$

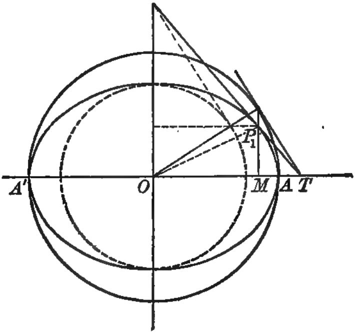


FIG. 211

The remarkable thing about this result is the fact that it is independent of b and of y_1. This means that if any other ellipse be given having the axis $A'A$ in common with the first ellipse, then the tangent drawn to this new ellipse at a point having the abscissa x_1 will also pass through T. This is therefore true of the circle drawn on $A'A$ as diameter. If $A'A$ is the major axis of the ellipse, this circle is called the ***major circle*** of the ellipse; similarly the circle drawn on the minor axis of any ellipse as diameter is called the ***minor circle***.

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* We do not in this article impose the restriction $a \geqq b$.

A geometric construction for the tangent at any point P_1 of an ellipse follows readily from the above considerations (assuming that in addition to the curve one of the axes is given). Figure 211 shows the construction using the major circle and, in broken lines, the construction using the minor circle.

The following theorem is of fundamental importance in discussing the geometric properties of the ellipse:

THEOREM 1. *The tangent and the normal to an ellipse at a given point bisect the angles formed by the focal radii drawn to the point.*

PROOF. We are to prove that the tangent at P_1 (Fig. 212) bisects the angle F_2P_1R, and that the normal at P_1 bisects the angle $F_1P_1F_2$. To this end we calculate first the tangent of the angle SP_1R. Using the equation of the ellipse as given above and taking the foci to be $F_2(c, 0)$ and $F_1(-c, 0)$, we have

$$\text{the slope of the tangent } P_1S = -\frac{b^2x_1}{a^2y_1},$$

$$\text{the slope of } F_1R \text{ (}i.e.\ F_1P_1\text{)} = \frac{y_1}{x_1+c}.$$

The tangent of the angle ϕ_1 from P_1S to P_1R is then

$$\tan\phi_1 = \frac{\dfrac{y_1}{x_1+c}+\dfrac{b^2x_1}{a^2y_1}}{1-\dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}}.$$

Simplifying this expression, we find

$$\tan\phi_1 = \frac{b^2}{cy_1}.$$

The tangent of the angle ϕ_2 from P_1S to P_1F_2 may evidently be obtained by simply changing c to $-c$ in the last result. (Why?) Hence,

$$\tan\phi_2 = -\frac{b^2}{cy_1}.$$

We conclude that $\phi_2 = -\phi_1$. This proves that P_1S bisects the angle F_2P_1R. That the normal bisects the angle $F_1P_1F_2$ follows at once from elementary geometry.

The theorem just proved leads at once to another geometric construction for the tangent (and normal) to an ellipse at a given point, supposing the foci of the ellipse are known.

THEOREM 2. *The foot of the perpendicular dropped from either focus on any tangent to an ellipse lies on the major circle.*

PROOF. (See Fig. 212.) Let S be the foot of the perpendicular dropped from F_2 on the tangent P_1S, and let it meet the line F_1P_1 in R. Then F_2P_1R is an isosceles triangle (why?) with $P_1R = P_1F_2$. We have then

$$F_1R = F_1P_1 + P_1F_2 = 2\,a. \quad (\S\ 225)$$

Also S is the mid-point of F_2R and O is the mid-point of F_1F_2. Hence $OS = \frac{1}{2}F_1R = a$, and S is on the major circle.

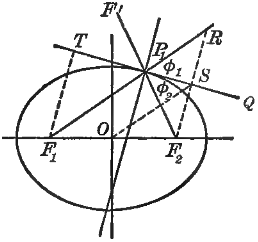

FIG. 212

We should note also that, if Q is any point on the tangent P_1S, then $QR = QF_2$, which is important in connection with the problem of drawing the tangents to an ellipse from an external point. (See Ex. 5, below.)

EXERCISES

1. Show how to construct the tangent to a given ellipse at a given point. (Two constructions, one using the major circle, one using the foci.)

2. Show that, in Fig. 211, OA is a mean proportional between OM and OT.

3. Show that, in Fig. 212, OF_1 is the mean proportional between the intercepts on the x-axis of the tangent and normal at P_1.

4. Prove analytically that S (Fig. 212) is on the major circle.

5. Show how to construct the tangents to an ellipse from a given external point Q. [HINT: Construct R (Fig. 212) as the intersection of two circles, one with center F_1 the other with center Q.]

6. Show that if a right angle moves with its vertex on a given circle and one of its sides passing through a fixed point within the circle, the other side will envelop an ellipse.

7. Use the result of Ex. 6 to construct a considerable number of tangents to an ellipse, given the major circle and one focus (the outline of the ellipse is not supposed to be given in advance, but will appear vividly after this problem is solved).

8. If an ellipse is rotated about its major axis the surface generated is called a ***prolate spheroid.*** Show that sound waves issuing from one focus will be reflected by the surface to the other focus. This principle is used in the so-called "whispering galleries."

9. By an argument similar to that used in § 212 show that the equation $xx_1/a^2 + yy_1/b^2 = 1$ is the equation of the line joining the points of contact of tangents drawn from (x_1, y_1) to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

240. Geometric Properties of the Hyperbola. Many of the geometric properties of the hyperbola are similar to corresponding properties of the ellipse, which is to be expected in view of the similarity of their equations. The following two theorems are fundamental:

THEOREM 1. *The tangent at any point of a hyperbola bisects the angle between the focal radii drawn to the point. The normal bisects the adjacent supplementary angle.*

THEOREM 2. *The foot of the perpendicular dropped from either focus on any tangent to a hyperbola is on the circle drawn on the transverse axis as diameter.*

The proofs of these theorems are left as exercises. They are similar to the proofs of the corresponding theorems on the ellipse. Draw figures illustrating Theorems 1 and 2.

Certain new properties of the hyperbola relating to the asymptotes will be found among the exercises below.

EXERCISES

1. Show how to construct the tangent and the normal to a given hyperbola at a given point.

2. If P is any point on a hyperbola, OA the semi-transverse axis, M the foot of the perpendicular dropped from P on OA (produced), and T the point in which the tangent at P meets OA, prove that OA is a mean proportional between OM and OT.

3. With the notation of Ex. 2 show that OF_1 is the mean proportional between ON and OT, F_1 being the focus on OA and N the point in which the normal at P meets OA (produced).

4. Prove Theorem 2 (§ 240) analytically.

5. Show how to construct the tangents to a hyperbola from an external point.

6. Show that if a right angle moves with its vertex on a given circle and one of its sides passing through a fixed point outside the circle the other side will envelop a hyperbola.

7. The construction of tangents to a hyperbola analogous to Ex. 7, p. 368.

8. Use Ex. 3 above and Ex. 3, p. 367, to show that an ellipse and hyperbola having the same foci intersect at right angles.

9. Prove that, if a tangent to a hyperbola meets the asymptotes in T_1 and T_2, the point of contact of the tangent is the mid-point of the segment T_1T_2.

10. Prove that the area of the triangle formed by any tangent and the asymptotes of a hyperbola is constant ($= ab$).

11. Show that if a straight line cuts a hyperbola in P_1 and P_2 and the asymptotes in Q_1 and Q_2 the segments P_1Q_1 and P_2Q_2 are equal. Use this result to construct any number of points of a hyperbola when the asymptotes and one point of the curve are given.

12. By an argument similar to that used in § 212 show that the equation $xx_1/a^2 - yy_1/b^2 = 1$ is the equation of the line joining the points of contact of the tangents drawn from (x_1, y_1) to the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

241. The Conics as Plane Sections of a Cone. We stated in § 216 that the ellipse, hyperbola, and parabola could all be obtained as the plane sections of a right circular cone. This we shall now proceed to prove. In doing so we shall get the machinery for solving problems of a more general type.

If a point P in a plane α (Fig. 213) is joined to a point S not in α by a straight line SP, the intersection P' of SP by a plane α' is called the ***projection*** of P from S upon α'. Similarly, if all the points of a curve C in α be joined to S, the intersections of these lines with a plane α' form a curve C', which is called the ***projection*** from S of the curve C. The point S is called the ***center of projection***, and the process described is called ***central projection***, to distinguish it from orthogonal projection previously considered (*e.g.* in § 135).

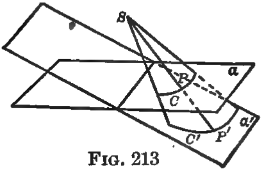

FIG. 213

If, now, the curve C in the plane α is a circle, the lines through S and the points of this circle form a cone with vertex S. This is not a right cone, in general. As the lines through S are not supposed to terminate in S, we get a so-called *complete* cone, or cone of two nappes, which consists of two congruent ordinary cones placed vertex to vertex so that their axes form a straight line. It will now be clear that a plane section of this cone is the same as the projection of the circle C from the vertex S upon the plane of section.

We have then reduced the problem to that of finding the central projection of a circle. We will solve it by finding the relation between the coördinates of a point P in α and the coördinates of the corresponding point P' in α'. To this end (Fig. 214) let O be the foot of the perpendicular dropped from S on the line of intersection of the planes α and α'. Let O be the origin and let the line of intersection OY of the two planes be the y-axis in the system of coördinates in each of the two planes. Let the line OX perpendicular to OY in α be the x-axis in α, and the line OX' perpendicular to OY in α' be the x-axis in α'. Let the angle between the two planes be θ; then $X'OX = \theta$. Now let $P(x, y)$ be any point in the plane α, and

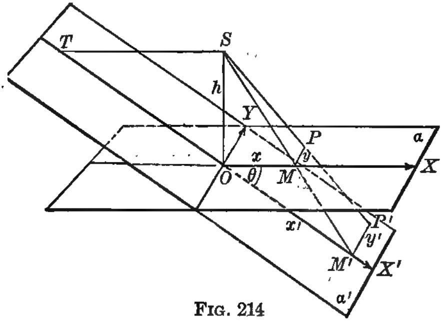

FIG. 214

let $P'(x', y')$ be the projection of P from S. We seek the relation connecting the coördinates x, y, x', y'.

Draw ST parallel to OX, and represent the length OS by h. Then we have

$$TO = \frac{h}{\sin\theta}, \quad TS = \frac{h}{\tan\theta}.$$

We then have from similar triangles

$$x' : (TO + x') = x : TS,$$

$$y' : y = SM' : SM = TM' : TO.$$

If we substitute the values of TO, TS, and TM' $(= TO + x')$, we obtain

$$\frac{x'}{x' + \dfrac{h}{\sin\theta}} = \frac{x}{\dfrac{h}{\tan\theta}},$$

and

$$\frac{y'}{y} = \frac{x' + \dfrac{h}{\sin\theta}}{\dfrac{h}{\sin\theta}}.$$

Solving these equations for x and y, respectively, we have

$$x = \frac{h\cos\theta\cdot x'}{\sin\theta\cdot x' + h}, \qquad y = \frac{hy'}{\sin\theta\cdot x' + h}.$$

If these expressions be substituted for x and y in the equation of any curve in the plane α, the resulting equation in x' and y' will be the equation of the projection of the curve in α'. To solve the problem we proposed at the outset, let the curve in the plane α be the circle

$$x^2 + y^2 = a^2.$$

The equation of the corresponding curve in α' is then

$$h^2\cos^2\theta\cdot x'^2 + h^2y'^2 = a^2\sin^2\theta\cdot x'^2 + 2\,ha^2\sin\theta\cdot x' + a^2h^2.$$

Collecting like terms, we have

$$(h^2\cos^2\theta - a^2\sin^2\theta)\,x'^2 + h^2y'^2 - 2\,ha^2\sin\theta\cdot x' - a^2h^2 = 0.$$

We see at once that this is the equation of a conic. It is an ellipse, a parabola, or a hyperbola according as

$$h^2\cos^2\theta - a^2\sin^2\theta > 0,\ = 0,\text{ or } < 0,$$

i.e. according as

$$\tan\theta - \frac{h}{a} < 0,\ = 0,\text{ or } > 0.$$

But h/a is the tangent of the angle ϕ which an element of the cone with vertex S makes with the plane α. If θ is less than this angle ϕ, the section of the cone is an ellipse; if θ is equal to ϕ, the section is a parabola; and if θ is greater than ϕ, the section is a hyperbola. Note that this result is in accordance with our geometric intuition of the situation.

EXERCISES

1. Prove that the central projection of *any* circle is a conic; that is, that a plane section (not through the vertex) of any circular cone (not necessarily a right cone) is a conic.

[HINT: Complete generality will be secured by taking the equation of the circle in α to be $x^2 + y^2 + dx + c = 0$. Why?]

2. Prove that the central projection of any conic is a conic.

3. Prove that the central projection of a straight line is a straight line.

4. Prove that there exists in α just one straight line which has no corresponding line in α', namely the line of intersection of α with the plane through S parallel to α'. This line is called the vanishing line of α.

5. Prove that the central projection of a circle in α is an ellipse, a parabola, or a hyperbola, according as the vanishing line in α meets the circle in no points, one point, or two points.

242. Poles and Polars. Diameters. We have had occasion in several exercises to note that the equation which represents the tangent to a conic *at* the point $P_1(x_1, y_1)$ when P_1 is on the curve, represents a straight line called the *polar* of P_1 when P_1 is any point in the plane. P_1 is then called the ***pole*** of the line with respect to the conic. The polar of a point *on* the conic is then the tangent at the point. We have also seen that the polar of a point P_1 through which pass two tangents to the conic is the line joining the two points of contact of the tangents. In the more extensive geometric theory of conics poles and polars play an important rôle.

A straight line passing through the center of an ellipse or hyperbola is called a ***diameter*** of the conic. Every diameter of an ellipse meets the curve in two points; some of the diameters of a hyperbola meet the curve in two points. These points are then called the extremities of the diameter, and the distance between them is called the length of the diameter. Any line parallel to the axis of a parabola is called a diameter of the parabola. Other properties are given in exercises below.

MISCELLANEOUS EXERCISES

PROPERTIES OF POLES AND POLARS

1. Write the equation of the polar of each of the following points with respect to the conic given, and draw the corresponding figure:

(*a*) $(1, 5)$; $2x^2 + y^2 = 4$. (*d*) $(-1, 3)$; $x^2 + y^2 + 4x - 6y - 2 = 0$.
(*b*) $(2, 0)$; $4x^2 - 9y^2 = 36$. (*c*) $(2, -3)$; $y^2 = 6x$.

2. Find the pole of the line $3x - 4y + 12 = 0$ with respect to the following conics:

(*a*) $3x^2 + 4y^2 = 12$; (*b*) $x^2 - 5y^2 = 20$; (*c*) $y^2 = 8x$; (*d*) $x^2 = 4y$.

3. Prove that in any conic the polar of a focus is the corresponding directrix.

4. Prove that in any conic, if P_1 and P_2 are two points such that the polar of P_1 passes through P_2, the polar of P_2 will pass through P_1.

5. From the result of the last exercise follows geometrically the following theorem: If a straight line be revolved about a point P and tangents are drawn at the points where it meets a conic, the locus of the intersection of these pairs of tangents is the polar of P with respect to the conic, or a part of the polar. Which part will it be?

6. Prove that the polar of any point on a directrix of a conic passes through the corresponding focus. [See Exs. 3 and 4.]

7. A straight line through a point P_1 meets a conic in C_1 and C_2, and the polar of P_1 in Q. Prove that P_1 and Q divide the segment C_1C_2 internally and externally in the same ratio.

[SOLUTION: Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points. Then the point P whose simple ratio with respect to P_1 and P_2 is λ has the coordinates

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}.$$

If these be substituted in the equation of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the resulting equation arranged as a quadratic in λ, we have

$$\left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - 1\right)\lambda^2 + 2\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1\right)\lambda + \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = 0.$$

The roots of this equation are the simple ratios of the points P_1 and P_2, respectively, with respect to the points C_1, C_2 in which the line P_1P_2 meets the ellipse. If the segment C_1, C_2 is to be divided internally and externally in the same ratio the roots λ_1, λ_2 of this equation must be equal numerically, but opposite in sign, *i.e.* $\lambda_1 + \lambda_2$ must be zero. The coefficient of λ in the above equation would then be zero, if the theorem to be proved is true. But the condition

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1=0,$$

is precisely the condition that $P_2(x_2, y_2)$ be on the polar,

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1,$$

of P_1 with respect to the ellipse. The similar proofs for the hyperbola and parabola are left as exercises.]

8. Two points P, Q on the line joining two given points C_1, C_2, are said to divide the segment C_1C_2, *harmonically*, if they divide the segment internally and externally in the same ratio (*i.e.* if $C_1P/PC_2 = -C_1Q/QC_2$). Show that the result of the last exercise leads to the following: The locus of a point Q, such that a given point P and the point Q divide harmonically the segment joining the points in which the line PQ meets a conic is the polar of P with respect to the conic, or a part of the polar. Which part is it? (Compare with the similar question in Ex. 5.)

PROPERTIES OF DIAMETERS

9. Prove that the locus of the mid-points of the chords of a conic drawn parallel to a given chord is a diameter of the conic.

[SOLUTION FOR THE ELLIPSE: Let the equation of the ellipse be $b^2x^2+a^2y^2=a^2b^2$, and let the slope of the given chord be m. Then any chord parallel to the given chord is $y=mx+k$. The abscissas x_1, x_2 of the points in which this chord meets the ellipse are the roots of the equation

$$(b^2+a^2m^2)x^2+2\,a^2mkx+a^2(k^2-b^2)=0.$$

The sum of the roots of this equation is

$$x_1+x_2=-\frac{2\,a^2mk}{b^2+a^2m^2}.$$

The coördinates (x', y') of the mid-point of the chord are then

$$x'=\tfrac{1}{2}(x_1+x_2)=-\frac{a^2mk}{b^2+a^2m^2},\quad y'=mx'+k=\frac{b^2k}{b^2+a^2m^2}.$$

The coördinates x', y' then satisfy the equation $y=-(b^2x)/(a^2m)$, no matter what the value of k is. The locus of the mid-points of the chords whose slope is m is, therefore, the straight line whose equation is $y=-(b^2x)/(a^2m)$. Since this straight line passes through the center of the ellipse, the theorem is proved for the ellipse. The similar proofs for the hyperbola and parabola are left as exercises.]

10. From the result of the last exercise, show how to construct a diameter of any conic, and hence (in case of ellipse and hyperbola) how to find the center, when only the outline of the curve is given.

11. Having given the outline of an ellipse or hyperbola, construct the center. Then show how to construct the principal axes (make use of the fact that the principal axes are axes of symmetry; a circle drawn with the center of the conic as center and suitable radius will meet the conic in the four vertices of a rectangle whose sides are parallel to the principal axes). Then construct the foci and the directrices.

12. Having given only the outline of a parabola, show how to construct the axis, the focus and the directrix.

13. Show that the tangent drawn to a conic at an extremity of a diameter is parallel to the chords which the diameter bisects.

14. If two diameters of a conic are such that each bisects the chords parallel to the other, the diameters are said to be ***conjugate***; and each is called the conjugate of the other. Prove that if m_1, m_2 are the slopes of two conjugate diameters of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, then we have $m_1m_2 = -b^2/a^2$.

15. Prove that, if m_1, m_2 are the slopes of two conjugate diameters of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, we have $m_1m_2 = b^2/a^2$.

16. The only conic for which all pairs of conjugate diameters are perpendicular is the circle.

17. The polars of the points on any diameter of an ellipse or hyperbola are parallel to the conjugate diameter.

18. If one extremity of a diameter of an ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has the coördinates (x_1, y_1), one extremity of the diameter conjugate to the given one will have the coördinates $(-y_1a/b, x_1b/a)$.

19. The area of a parallelogram circumscribed about an ellipse whose sides are parallel to two conjugate diameters is constant and equal to $4ab$.

20. Prove that, if a_1 and b_1 are the lengths of two conjugate semi-diameters of an ellipse, $a_1^2 + b_1^2 = a^2 + b^2$.

21. Prove that any pair of conjugate diameters of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ are also conjugate diameters of $b^2x^2 - a^2y^2 = -a^2b^2$.

22. If a diameter of a hyperbola with center O meets the hyperbola in P and the conjugate diameter meets the conjugate hyperbola in Q, prove that $OP^2 - OQ^2 = a^2 - b^2$.

CHAPTER XIV

POLAR COÖRDINATES

243. Review. Polar coördinates, introduced in §§ 112–114, are often useful in studying geometry analytically. The present chapter is devoted to illustrating some of the principles involved and their applications.

EXERCISES

1. What is the locus of points for which ρ is constant?

2. What is the locus of points for which θ is constant?

3. Show that the points (ρ, θ) and $(\rho, -\theta)$ are symmetric with respect to the polar axis.

4. Show that the points (ρ, θ) and $(-\rho, \theta)$ are symmetric with respect to the pole.

5. Show that the points (ρ, θ) and $(\rho, \theta + 180°)$ are symmetric with respect to the pole.

6. Find the distance between the points $A(2, 45°)$ and $B(7, 105°)$.

[HINT. Use the law of cosines.]

7. Prove that the distance between the points (ρ_1, θ_1) and (ρ_2, θ_2) is

$$\sqrt{\rho_1^2 + \rho_2^2 - 2\,\rho_1\rho_2 \cos(\theta_2 - \theta_1)}.$$

244. Locus of an Equation. The locus of an equation in the variables ρ and θ is such that:

(1) Every point whose coördinates (ρ, θ) satisfy the equation is on the locus or curve, and

(2) A set of coördinates* of every point on the locus or curve satisfies the equation.

---

* Not necessarily every set. Thus, the point $(2, 60°) = (-2, 240°)$ is on the locus of $\rho = 1 + 2\cos\theta$; but the second set of coördinates does not satisfy the equation.

The curve may be sketched by computing a table of corresponding values of ρ and θ, plotting the corresponding points, and then sketching the curve through them. The amount of work may often be shortened if one makes use of the following obvious rules for symmetry:

If a polar equation is left ***unchanged,***

(a) *when* θ *is replaced by* $-\theta$, *the locus is symmetric with respect to the polar axis.*

(b) *when* ρ *is replaced by* $-\rho$, *the locus is symmetric with respect to the pole.*

(c) *when* θ *is replaced by* $180° + \theta$, *the locus is symmetric with respect to the pole.*

(d) *when* θ *is replaced by* $180° - \theta$, *the locus is symmetric with respect to the line through the pole perpendicular to the polar axis.*

It should be borne in mind, however, that none of these rules are *necessary* conditions for symmetry. Why not?

245. Illustrative Examples.

We shall illustrate the methods of plotting curves in polar coördinates by the following examples.

EXAMPLE 1. *Discuss and plot the locus of the equation* $\rho = 4 \cos \theta$.

The locus is symmetric with respect to the polar axis. If we plot points from 0° to 90°, we obtain the upper half of Fig. 215. Then by symmetry we obtain the complete graph given in Fig. 215. Why?

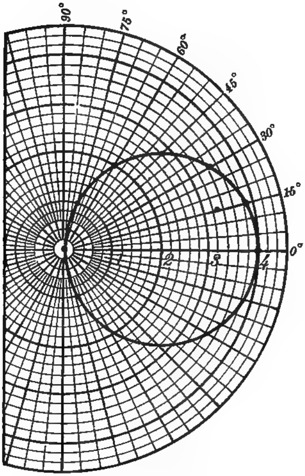

FIG. 215

θ	0°	30°	45°	60°	90°
ρ	4	3.5	2.8	2	0

Example 2. *Discuss and plot the locus of the equation* $\rho = 4 \sin^2 \theta$.

The locus is symmetric with respect to the pole, the polar axis, and the line through the pole perpendicular to the polar axis. If we plot points in the range $\theta = 0°$ to $\theta = 90°$, and make use of symmetry, we have the complete figure which is given in Fig. 216.

θ	0°	30°	45°	60°	90°
ρ	0	1	2	3	4

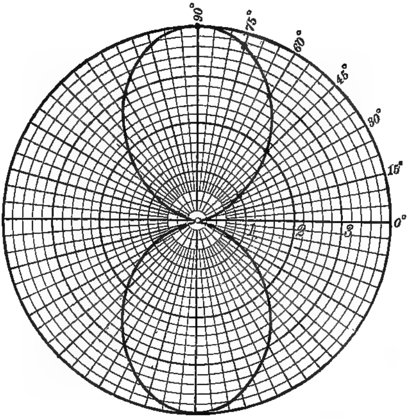

Fig. 216

The branches constructed by symmetry should be checked by substituting in the original equation the coördinates of at least one point on each branch. Serious errors may thus be avoided.

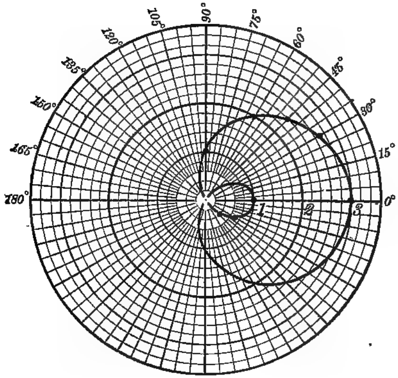

Fig. 217

Example 3. *Discuss and plot the locus of the equation* $\rho = 1 + 2 \cos \theta$.

The curve is symmetric with respect to the polar axis. If we plot points from $\theta = 0°$ to $\theta = 180°$, we get the points shown in Fig. 217. Then by symmetry we get the complete graph, or the curve in Fig. 217.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
ρ	3	$1+\sqrt{3}$	$1+\sqrt{2}$	2	1	0	$1-\sqrt{2}$	$1-\sqrt{3}$	-1

EXERCISES

Are the following loci symmetric with respect to the pole? The polar axis? The line through the pole perpendicular to the polar axis?

1. $\rho = a \cos \theta.$
2. $\rho = a \sin \theta.$
3. $\rho = a\,(1 - \cos \theta).$
4. $\rho = a\,(1 - \sin \theta).$
5. $\rho^2 = a \cos 2\,\theta.$
6. $\rho^2 = a \sin 2\,\theta.$
7. $\rho = a \cos 2\,\theta.$
8. $\rho = a \sin 2\,\theta.$
9. $\rho = a \sin^2 \theta.$
10. $\rho = \sin^2 \dfrac{\theta}{2}.$
11. $\rho = \theta.$
12. $\rho^2 \cos \theta = 4.$

Discuss and plot the locus of each of the following equations.

13. $\rho = 5.$
14. $\rho = -5.$
15. $\rho^2 = 25.$
16. $\theta = 30°.$
17. $\theta = -30°.$
18. $\rho = 8 \cos \theta.$
19. $\rho = -8 \cos \theta.$
20. $\rho = 8 \sin \theta.$
21. $\rho = -8 \sin \theta.$
22. $\rho \cos \theta = 4.$
23. $\rho \cos \theta = -4.$
24. $\rho \sin \theta = 5.$
25. $\rho \sin \theta = -5.$
26. $\rho = 1 - \cos \theta.$
27. $\rho = 1 + \cos \theta.$
28. $\rho = 1 - \sin \theta.$
29. $\rho = 1 + \sin \theta.$
30. $\rho = 1 + 3 \cos \theta.$
31. $\rho = 1 + 2 \sin \theta.$
32. $\rho = 1 - 2 \cos \theta.$
33. $\rho = 1 - 2 \sin \theta.$
34. $\rho = 2 + \cos \theta.$
35. $\rho = 2 + \sin \theta.$
36. $\rho = 4 \tan \theta.$
37. $\rho = \dfrac{1}{1 - \cos \theta}.$

246. Standard Equations. We shall now derive polar equations for the straight line and the conic sections.

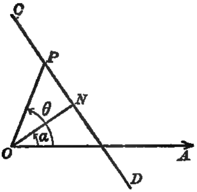

FIG. 218

The straight line. Let CD be any straight line (Fig. 218) $ON = p$ the perpendicular let fall upon it from the pole O, and α the angle which this perpendicular makes with the polar axis. Let (ρ, θ) be any point on the line.

Then
$$\frac{ON}{OP} = \cos(\theta - \alpha) \quad \text{or} \quad \cos(\alpha - \theta).$$

But by § 120,
$$\cos(\theta - \alpha) = \cos(\alpha - \theta).$$

Hence

$$(1) \qquad p = \rho \cos(\theta - \alpha)$$

is the desired equation.

If the line is perpendicular to the polar axis, its equation is $\rho \cos \theta = p$. Why?

If the line is parallel to the polar axis, its equation is $\rho \sin \theta = p$. Why?

The circle. Let $C\ (c, \alpha)$ be the center of a circle of radius r and $P\ (\rho, \theta)$ any point on the curve (Fig. 219). In the triangle OCP, $OC = c$, $OP = \rho$ and the angle $COP = \pm (\theta - \alpha)$ depending upon the position of the point P. But since $\cos (\theta - \alpha) = \cos (\alpha - \theta)$, we have from the law of cosines, § 126,

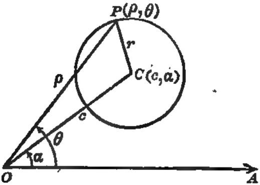

FIG. 219

$$r^2 = c^2 + \rho^2 - 2\, c\, \rho \cos (\theta - \alpha) \tag{2}$$

as the equation of the desired locus.

If the center C is upon the polar axis ($\alpha = n\pi$), equation (2) becomes

$$r^2 = c^2 + \rho^2 \pm 2\, c\, \rho \cos \theta. \tag{3}$$

If the circle passes through the pole ($r = \pm\ c$), equation (2) becomes

$$\rho = \pm\ 2\, r \cos (\theta - \alpha). \tag{4}$$

If the center C is upon the polar axis ($\alpha = 0$) and the circle passes through the pole ($c = \pm\ r$), equation (2) becomes

$$\rho = \pm\ 2\, r \cos \theta. \tag{5}$$

If the center is at the pole ($c = 0$), equation (2) becomes $\rho = \pm r$.

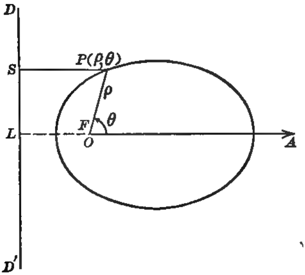

FIG. 220

The Polar Equation of any Conic. The polar equation of any conic may now be derived. Let DD' be the directrix, F the corresponding focus, and e the eccentricity. Let the perpendicular through F to DD' meet DD' in L. Let the segment $LF = p$, and take F as the pole and the extension of the line LF as the polar axis. If $P(\rho, \theta)$ is any point on the curve and PS is the perpendicular from P to DD', then by the definition of a conic, we have

$$FP = e \cdot PS,$$

that is,

$$\rho = e(p + \rho\cos\theta),$$

or

$$\rho = \frac{ep}{1 - e\cos\theta} \tag{6}$$

which is the polar equation of a conic. If $e < 1$, the equation represents an ellipse; if $e = 1$, a parabola; $e > 1$, a hyperbola.

EXERCISES

1. Derive the equation $\rho = 2r\cos\theta$ [(5), § 246] directly from a figure.

2. Derive the polar equation of the ellipse assuming the right-hand focus as the pole and the major axis as the polar axis.

3. Derive the polar equation of a hyperbola assuming the right-hand focus as the pole and the transverse axis as the polar axis.

4. Derive the polar equation of the circle which passes through $(0, 0°)$ and has its center at $(r, 90°)$; $(r, 270°)$.

5. Derive the polar equation of the parabola assuming the focus at the pole and the directrix the line $\rho\sin\theta = p$; $\rho\sin\theta = -p$.

6. The difference of the focal radii of a certain hyperbola is 3, and the distance between the foci is 6. Find a polar equation of the curve.

247. Other Curves. What is the advantage of polar coördinates? Why not continue to use only rectangular coördinates? The answer to these questions is that in certain kinds of problems polar coördinates are much more convenient. The following examples will illustrate the desirability of polar coördinates.

The limaçon. Through a fixed point O upon any given circle of radius a, a chord OP_1 is drawn and produced to P so that $P_1P = k$, where k is a given constant (Fig. 221). Find the locus of P* as P_1 describes the circle.

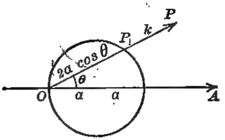

FIG. 221

If $\rho = 2\,a\cos\theta$ is the equation of the circle and the pole is the fixed point, then the locus of P is

$$(7) \qquad \rho = 2\,a\cos\theta + k.$$

If $k = 2\,a$, the equation may be written in the form

$$(8) \qquad \rho = 2\,a(1 + \cos\theta)$$

and the curve is known as the ***cardioid***, on account of its heart-shaped form (Fig. 222).

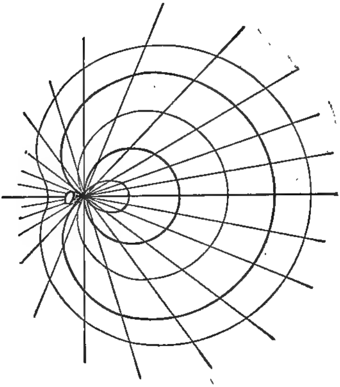

FIG. 222. — The Limaçon

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* This curve is known as the *limaçon of Pascal*. It was invented by BLAISE PASCAL (1623–1662), a famous French mathematician and philosopher. The word *limaçon* means *snail*. The Germans call this curve *die Pascal'sche Schnecke*.

The cissoid. OA is a fixed diameter of a fixed circle (Fig. 223). At the point A a tangent is drawn, while about the point O a secant revolves which meets the tangent in B, and the circle in C. Find the locus of a point P on OB so determined that $OP = CB$.

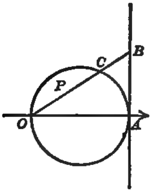

FIG. 223

Take O as the pole and OA as the polar axis of a system of polar coördinates. If we denote OA by $2a$, then the equation of the circle is $\rho = 2a\cos\theta$. Let P be denoted by (ρ, θ). Now

$$\rho = OP = OB - PB.$$

But

$$OB = 2a\sec\theta \text{ and } PB = OC = 2a\cos\theta.$$

Therefore $$\rho = 2a(\sec\theta - \cos\theta),$$

or

(9) $$\rho = 2a\tan\theta\sin\theta.$$

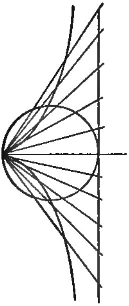

FIG. 224. — The Cissoid

The locus of this equation is given in Fig. 224. The curve is known as the *cissoid of Diocles.*[*]

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[*] *Cissoid* (Greek, κισσός = ivy) means *ivy-like.* The Greeks considered only the part of the curve lying within the circle. DIOCLES was a Greek mathematician who lived sometime between 217 B.C. and 70 B.C. By means of this curve, Diocles showed how to construct the side of a cube whose volume is twice the volume of a given cube. See Ex. 4, p. 388.

Conchoid of Nicomedes.*[*]* A straight line revolves about a fixed point O and meets a fixed straight line MN in the point Q. From Q a fixed length is laid off on OQ in both directions. The locus of the two points, P and P', thus determined is called a conchoid.

Let O be the pole and the line OR through O perpendicular to MN be the polar axis of a system of polar coördinates

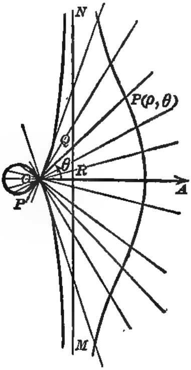

FIG. 225. — The Conchoid

Let (ρ, θ) be the coördinates of any position of the generating point P (or P'). Then

$$\rho = OP (\text{or } OP') = OQ \pm QP = OR \sec\theta \pm QP.$$

But OR and QP are given constants; call them a and b respectively. Then

$$(10) \qquad \rho = a\sec\theta \pm b$$

is the desired equation of the conchoid.

---

[*] *Conchoid* (Greek, κογχος = mussel) means *mussel-like*. NICOMEDES was a contemporary of Diocles. He invented the conchoid for the purpose of trisecting an angle, which is one of the famous problems of antiquity. This problem cannot be solved by means of the compass and straightedge alone.

248. Spiral Curves. A spiral is a curve traced by a point which revolves about a fixed point called the center, but continually recedes from or continually approaches the center according to some definite law.

The ***spiral of Archimedes*** is the locus of a point such that its radius vector is proportional to its vectorial angle. Therefore its equation is

$$\rho = k\theta, \tag{11}$$

where k is a constant.*

The form of the equation shows that the locus passes through the pole, and that the radius vector increases without limit as the number of revolutions increases without limit. Figure 226 represents a portion of the locus for $k = \frac{1}{75}$, with θ expressed in degrees.

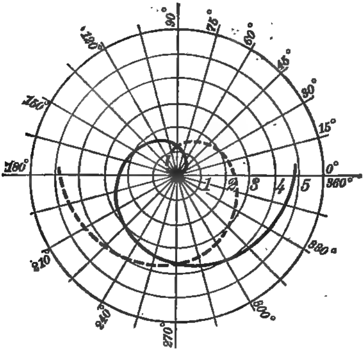

FIG. 226

The ***hyberbolic or reciprocal spiral*** is the locus of a point such that its radius vector is inversely proportional to its vectorial angle. The equation of the locus is therefore

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* In this example, and in those that follow, it is usual to express the angle θ in radians; but this is not necessary, since the same result can be obtained by choosing a different value for k if θ is expressed in degrees.

(12) $$\rho = \frac{k}{\theta},$$

where k is a constant. Figure 227 represents a portion of the graph for $k = 70$ and for positive values of θ (expressed in degrees).

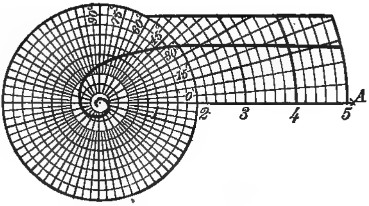


FIG. 227

The ***logarithmic spiral*** is the locus of a point such that the logarithm of its radius vector is proportional to its vectorial angle, *i.e.*

(13) $$\log \rho = k\theta,$$

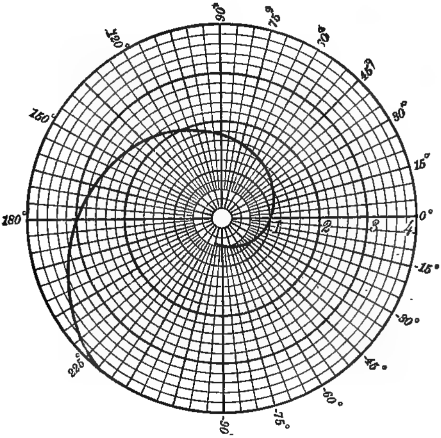


FIG. 228

where k is a constant. If the base of the system of logarithms is b, the equation may be written in the form $\rho = b^{k\theta}$. Figure 228 represents a portion of this locus when $b = 3$, for $k = \frac{1}{180}$, with θ expressed in degrees.

EXERCISES

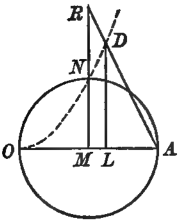
1. Discuss the form of the limaçon (7), when $|k| < 2a$. When $|k| > 2a$.

2. Solve the locus problem used to define the limaçon by means of rectangular coördinates, and compare the merits of the two solutions.

3. By taking the line OA (Fig. 223) as the x-axis and the tangent to the circle at O as the y-axis, prove that the equation of the cissoid is

$$y^2 = \frac{x^3}{2a - x}.$$

4. Duplication of a Cube. In the adjoining figure, let $MN = a$ and $MR = 2a$. Draw RA and let it meet the cissoid in the point D whose ordinate is LD. Prove that $LD^3 = 2\,OL^3$. If $MR = n \cdot a$ prove that $LD^3 = n \cdot OL^3$.

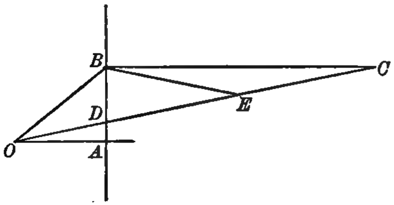


5. If in Fig. 225 the line MN is taken as the x-axis and the line ORA as the y-axis, prove that the equation of the conchoid is

$$x^2y^2 = (y + a)^2(b^2 - y^2).$$

Compare the merits of this solution with that on p. 385.

6. ***Trisection of an angle.*** Let AOB be the angle to be trisected. Through a convenient point A on one side OA of the angle draw AB perpendicular to OA. Through B draw a line BC parallel to OA. From O as fixed point, and AB as fixed line, and $2 \cdot OB$ as a constant distance, describe a conchoid meeting BC in C. Angle AOC is then $\frac{1}{3}$ AOB.



[HINT: E is the mid-point of DC; then $OB = BE = EC$. The result then follows from elementary geometry.]

7. Show that in the conchoid, if

(a) $b > a$, the curve has an oval at the left, as in Fig. 225;

(b) $b = a$, the oval closes up to a point;

(c) $b < a$, there is no oval and both branches lie to the right of the point O.

8. Draw the *parabolic spiral*, which is defined by the equation $\rho^2 = k\theta$. Take $k = \frac{1}{10}$ with θ in degrees, and use only the positive values of ρ.

9. Draw the *lituus*, which is defined by the equation $\rho^2 = k/\theta$. Take $k = 90$ with θ in degrees, and use only the positive values of ρ.

249. Relation between Rectangular and Polar Coördinates. Take O the origin of a system of rectangular axes as the pole, and the positive half of the x-axis as the polar axis, of a system of polar coördinates.

Let (x, y) and (ρ, θ) be respectively the rectangular and polar coördinates of any point P. Then $x/\rho = \cos\theta$, $y/\rho = \sin\theta$. Hence, we have

$$(14) \qquad \begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta. \end{cases}$$

It is here assumed that the coördinates of P are so chosen that $OP = \rho$ and angle $XOP = \theta$. This is always possible. If ρ is positive, x always has the sign of $\cos\theta$ and y the sign of $\sin\theta$.

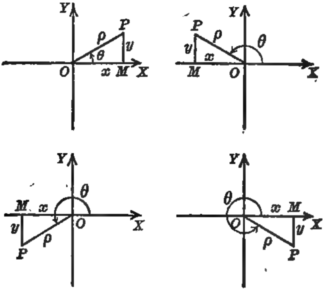


FIG. 229

Conversely, if ρ is positive, we see from Fig. 229 that

$$(15) \qquad \begin{cases} \rho^2 = x^2 + y^2, & \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}, \\ \theta = \text{arc}\tan\left(\dfrac{y}{x}\right), & \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}. \end{cases}$$

EXERCISES

Transform the following equations into equations in rectangular coördinates. State in each case whether the graph is easier to sketch from the polar or the rectangular equation.

1. $\rho = 1 - \cos\theta$.

2. $\rho = 1 + \sin\theta$.

3. $\rho = 4$.

4. $\rho\cos\theta = 5$.

5. $\rho\sin\theta = -2$.

6. $\rho^2\sin 2\theta = 3$.

7. $\rho^2\cos 2\theta = 4$.

8. $\rho^2 = \cos 2\theta$.

9. $\rho = \theta$.

10. $\rho = \frac{1}{\theta}$.

11. $\rho^2 = \theta$.

12. $\rho^2 = \frac{1}{\theta}$.

13. $\rho = a\sec\theta + b$.

14. $\rho = 2a\sec\theta\tan\theta$.

15. $\rho = 4\cos 2\theta$.

Transform the following equations into equations in polar coördinates:

16. $x^2 + y^2 = 4x$.

17. $(x^2 + y^2)^2 = x^2 - y^2$.

18. $x - y = 0$.

19. $y^2 = 4x$.

20. $9x^2 + 4y^2 = 36$.

21. $xy = 4$.

22. $x\cos\alpha + y\sin\alpha = p$.

23. $(y^2 + x^2 - 2x)^2 = x^2 + y^2$.

24. $x^3 = y^2(2 - x)$.

25. $x^2y^2 = (y + 2)^2(4 - y^2)$.

MISCELLANEOUS EXERCISES

Sketch the following curves:

1. $\rho = a\cos 2\theta$.

2. $\rho = a\cos 3\theta$.

3. $\rho = a\sin 2\theta$.

4. $\rho = a\sin 3\theta$.

5. $\rho = a\cos 4\theta$.

6. $\rho = a\sin 4\theta$.

7. $\rho = a\cos 5\theta$.

8. $\rho = a\sin 5\theta$.

9 $\rho = a\sin\frac{\theta}{2}$.

10. $\rho = a\cos\frac{\theta}{2}$.

Find the points of intersection of the following pairs of curves. Plot the curves in each case and mark with their respective coördinates the points of intersection.

11. $\rho = 1 + \cos\theta$,
$4(1 + \cos\theta)\rho = 1$.

12. $\rho = 4$,
$\rho\cos\theta = 2$.

13. $\rho = \sqrt{2}$,
$\rho = 2\sin\theta$.

14. $\rho = 1 + \cos\theta$,
$2\rho = 3$.

15. $\rho = 2(1 - \sin\theta)$,
$(1 + \sin\theta)\rho = 1$.

16. $\rho = \cos\theta$,
$\rho = 1 + 2\cos\theta$.

Solve the following exercises by the use of polar coördinates:

17. Find the locus of the center of a circle which passes through a fixed point O and has a radius 2.

18. Prove that if from any point O a secant is drawn cutting a circle in the points P and Q, then $OP \cdot OQ$ is constant for all positions of the secant.

[HINT. By using equation (2), § 246, show that the product of the roots is constant.]

19. Secants are drawn to a circle through a fixed point O on the circumference. Find the locus of the middle points of their chord segments.

20. Find the locus of the middle points of the focal radii issuing from one focus of an ellipse; parabola; hyperbola.

21. The focal radii of a parabola are produced a constant length. Find the locus of their end-points.

22. Through a fixed point O on a fixed circle a variable secant OP is drawn cutting the circle in R. If $RP = 3\ OR$, find the locus of P.

CHAPTER XV

PARAMETRIC EQUATIONS

250. Parametric Equations. As a point $P(x, y)$ moves along a given curve, the x- and y- coördinates of the point vary. So do many other quantities connected with this point, as for example, in general, its distance OP from the origin, the angle θ which OP makes with the x-axis, its distance from a fixed line, etc. It is sometimes convenient to express x and y in terms of one of these variables. This third variable, in terms of which the variables x and y are expressed, is called a ***parameter.*** For example, we see that the coördinates of any point $P(x, y)$ on the circle whose center is at the origin and whose radius is r, can be expressed in the form

$$(1) \qquad \begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases}$$

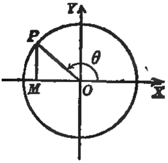


FIG. 230

where θ is the angle XOP (Fig. 230). These are then ***parametric equations of the circle.*** If we eliminate the parameter θ between these two equations by squaring and adding them, we obtain the equation

$$x^2 + y^2 = r^2,$$

which is the rectangular equation of the circle.

Similarly, a pair of ***parametric equations of the ellipse***

(2) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are

(3) $$\begin{cases} x = a\cos\theta, \\ y = b\sin\theta\,; \end{cases}$$

for, these values of x and y are seen to satisfy equation (2) for all values of θ.

The geometric interpretation of equations (3) is important. In Fig. 231, a geometric construction given in § 226 is used.

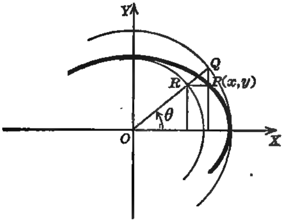


FIG. 231

The abscissa of P is equal to the abscissa of Q, *i.e.* $a\cos\theta$, the ordinate of P is equal to the ordinate of R, *i.e.* $b\sin\theta$. Therefore the coördinates of P are $x = a\cos\theta$, $y = b\sin\theta$. The angle θ is known as the ***eccentric angle*** of the point P. We should note that θ is not the angle XOP.

A pair of ***parametric equations for the hyperbola***

(4) $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

are

(5) $$x = a\sec\theta,\ y = b\tan\theta,$$

for these values of x and y satisfy the equation (4).

It is important to note that a given curve may have as many sets of parametric equations as we please. For example, parametric equations of a circle may be written in the form

$$x = a \cos t, \; y = a \sin t,$$

as above; or they may be written in the form

$$x = \frac{2\,az}{1+z^2}, \quad y = \frac{a\,(1-z^2)}{1+z^2},$$

or in any one of many other forms.

EXERCISES

1. Show that $x = t$, $y = 2 - t$ are parametric equations of a straight line.

2. Show that $x = \frac{1}{2}\,pt^2$, $y = pt$ are a pair of parametric equations of the parabola $y^2 = 2\,px$.

3. Write two pairs of parametric equations for the line $y = x$.

4. Prove that
$$x = \frac{a\,(1-t^2)}{1+t^2}, \quad y = \frac{2\,at}{1+t^2}$$
are parametric equations of a circle.

5. Write a pair of parametric equations for the rectangular hyperbola $x^2 - y^2 = a^2$.

6. Show that $x = A \cos\theta + B \sin\theta$, $y = A \sin\theta - B \cos\theta$ are parametric equations of a circle.

7. Prove that $x = 6 + 4\cos\theta$, $y = -2 + 3\sin\theta$ are parametric equations of an ellipse.

8. Write a pair of parametric equations for the circle
$$(x-a)^2 + (y-b)^2 = r^2.$$

9. Prove that $x = 6 + 4\sec\theta$, $y = -2 + 3\tan\theta$ are parametric equations of a hyperbola.

10. Find the equation of the tangent to

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, at $x_1 = a\cos\theta_1$, $y_1 = b\sin\theta_1$.

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, at $x_1 = a\sec\theta_1$, $y_1 = b\tan\theta_1$.

(c) $y^2 = 2\,px$, at $x_1 = \frac{1}{2}\,pt_1^2$, $y_1 = pt_1$.

11. Prove that the tangents to $y^2 = 2px$ at $(\frac{1}{2}pt_1^2, pt_1)$, $(\frac{1}{2}pt_2^2, pt_2)$ meet at the point $[\frac{1}{2}pt_1t_2, \frac{1}{2}p(t_1 + t_2)]$.

12. Write the equation of the tangent to $y^2 = 4ax$ at $x_1 = \frac{a}{m_1^2}$, $y_1 = \frac{2a}{m_1}$.

13. Show that

$$x = \frac{3at}{1 + t^3},\quad y = \frac{3at^2}{1 + t^3}$$

are parametric equations of the curve $x^3 + y^3 = 3axy$.

14. Show that $x = a\cos^3\theta$, $y = a\sin^3\theta$ are parametric equations of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

15. Find the x- and y-equations of the curve whose parametric equations are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

251. Sketching Loci of Parametric Equations. If we assign a series of values to the parameter and determine the series of corresponding pairs of values for x and y, we can interpret these values as the coördinates of points on a curve. Plotting these points and sketching a curve through them, we have the graph of the curve whose parametric equations were given.

EXAMPLE. A pair of parametric equations giving the path of a body projected horizontally from a height of 400 ft. with a velocity of 10 ft./sec., are $x = 10t$, $y = 400 - 16t^2$. Sketch the locus.

t	0	1	2	3	4	5
x	0	10	20	30	40	50
y	400	384	336	256	144	0

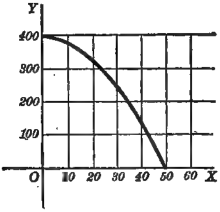


FIG. 232

In the preceding table are given the values of x and y corresponding to the integral values of t from 0 to 5 inclusive. Plotting these points we have the graph in Fig. 232.

This curve is of course the same that we should obtain by first eliminating t and then plotting from the equation in x and y.

252. The Time as Parameter. Suppose a point moves in a plane. At every instant of time t the point occupies a certain position (x, y). In other words, the coördinates x and y of the point P are functions of t, *i.e.*

$$\left\{\begin{aligned} x &= \text{a function of } t, \\ y &= \text{a function of } t. \end{aligned}\right. \tag{6}$$

These equations are then parametric equations of the path traversed by the point.

Such equations arise frequently in mechanics when it is desired to describe the motion of a body subject to various forces. For example, if a body is projected from a point O $(0, 0)$ in a vertical plane at time $t = 0$, with an initial velocity v_0, and making an angle α with the horizontal (x-axis), its position at the end of t seconds* is given by the equations

$$\left\{\begin{aligned} x &= tv_0 \cos \alpha, \\ y &= tv_0 \sin \alpha - \tfrac{1}{2} gt^2, \end{aligned}\right. \tag{7}$$

where, if v_0 is measured in ft./sec., g is a constant approximately equal to 32.2. The use of these *equations of a projectile* will be illustrated in the next article and the exercises following it.

EXERCISES

Sketch the following curves from their parametric equations.

1. $x = t,$
$y = t + 2.$

2. $x = r^2,$
$y = r.$

3. $x = s + 1,$
$y = s^2.$

4. $x = t^2 - 1,$
$y = t^2 - 1.$

5. $x = z^2 + 1,$
$y = z.$

6. $x = t,$
$y = \dfrac{1}{t}.$

7. $x = t,$
$y = -\dfrac{1}{t}.$

8. $x = t^3,$
$y = t^2.$

9. $x = t^2 + 1,$
$y = t^3 - 1.$

10. $x = t,$
$y = t - t^3.$

11. $x = \sin \theta,$
$y = \cos \theta.$

12. $x = \tan \theta,$
$y = \sec \theta.$

13. $x = 10\, t \cos 30°,$
$y = 25 + t \sin 30° - 16\, t^2.$

14. $x = 5\, t,$
$y = 30 - 16\, t^2.$

---

* The resistance of the air being neglected.

253. The Path of a Projectile in Vacuo. The equations (7) given in § 252 yield many results of interest regarding the paths of projectiles. Some of these are given in this article and others are found in the exercises below. They are, of course, only approximations to the actual behavior of a projectile, in view of the fact that the resistance of the air has been neglected.*

By eliminating t between the equations (7), § 252, we obtain the equation of the path in rectangular coördinates (x, y):

$$y = x \tan \alpha - \frac{gx^2}{2\,v_0^2} \sec^2 \alpha. \tag{8}$$

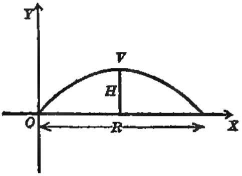


Fig. 233

The path is, therefore, *a parabola*, with a vertical axis. The vertex of the parabola is at the point (Fig. 233)

$$V = \left(\frac{v_0^2}{2\,g} \sin 2\,\alpha,\ \frac{v_0^2 \sin^2 \alpha}{2\,g}\right). \tag{9}$$

The greatest height above the horizontal is

$$H = \frac{v_0^2 \sin^2 \alpha}{2\,g}. \tag{10}$$

The complete range, *i.e.* the distance from O to the point where the projectile again meets the horizontal, is found as follows:

If in (7) we place $y = 0$, we find $t = 0$ and

$$t = 2(v_0/g) \sin \alpha.$$

The value of x for the second value of t found is the desired range R, *i.e.*

$$R = \frac{v_0^2}{g} \sin 2\,\alpha.$$

This result could also have been found by placing $y = 0$ in (8) and solving for x. Why?

---

* For the theoretical and practical discussion of the flight of actual projectiles (whose motion is appreciably affected by the resistance of the air) the student is referred to Alger, *The Groundwork of Naval Gunnery, or External Ballistics.*

EXERCISES

1. A gun is fired at an elevation of 30°. Find the range if the muzzle velocity of the shell is 1000 ft./sec. *Ans.* About 5 mi.

2. What is the greatest height reached by the projectile in Ex. 1? How long is its time of flight?

3. What must be the initial velocity of a baseball thrown at an angle of 45° in order that it may travel 200 ft. before hitting the ground?

4. A stone is thrown from a tower 100 ft. high, with an angular elevation of 45° and an initial velocity of 64 ft./sec. How far from the foot of the tower will the stone hit the ground?

5. The great pyramid of Cheops is 450 ft. high. Its base is a square 746 ft. on a side. A ball is thrown upwards from the top in a direction making an angle of 20° with the horizontal and with the velocity of 80 ft./sec. Will the ball clear the base of the pyramid?

6. Prove that for a given initial angle of elevation the range of a projectile is proportional to the square of the initial velocity.

7. Prove that for a given initial velocity the maximum range is obtained when the angle of elevation is 45°.

8. Prove that with the notation of § 253, the time of flight of a projectile from O to (x, y) is $(x/v_0)\sec\alpha$; from O to $(R, 0)$ is $(2v_0/g)\sin\alpha$.

9. Prove that the paths of a projectile with given v_0, but varying α, have the same directrix.

10. Prove that the coördinates of the focus of the path of a projectile are

$$\left(\frac{v_0^2 \sin 2\alpha}{2g}, \frac{-v_0^2 \cos 2\alpha}{2g}\right).$$

Hence show that the locus of the foci of all paths in a given vertical plane with the same v_0 is a circle with center at O.

11. Prove that the parabola of maximum range has its focus on the x-axis.

12. Prove that the locus of the vertices of the paths with given v_0 is an arc of an ellipse.

13. Prove that the locus of the vertices of the paths with a given α and a varying v_0 is a straight line.

14. Prove that the locus of the foci of the paths with a given α and a varying v_0 is a straight line.

254. Locus Problems. Parametric equations of a curve are sometimes much more easily obtained and easier to work with than either the rectangular or polar equations. The following problems illustrate some of the methods that may be employed.

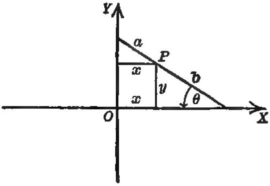


FIG. 234

EXAMPLE 1. A line of fixed length moves so that its ends always remain on the coördinate axes. Find the locus generated by any point of the line.

Call the point whose locus is desired $P(x, y)$. Since the line is of fixed length, call the segments into which P divides it, a and b (Fig. 234). Then $x = a \cos \theta$, $y = b \sin \theta$. Therefore the locus of P is an ellipse (§ 250).

EXAMPLE 2. Find the locus of a point P on a circle which rolls along a fixed line.

Take for origin the point O where the moving point P touches the fixed line. If r is the radius of the circle and the angle PCD (Fig. 235) is θ radians, then $PD = r \sin \theta$, $DC = r \cos \theta$ and $OB = \text{arc } BP = r\theta$.

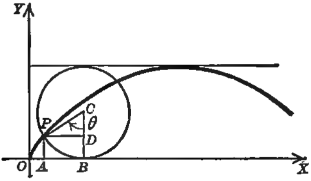


FIG. 235

Now if P is denoted by the coördinates (x, y),

$$x = OA = OB - AB = OB - PD = r\theta - r \sin \theta = r(\theta - \sin \theta),$$
$$y = AP = BC - DC = r - r \cos \theta = r(1 - \cos \theta).$$

Therefore

$$(11) \qquad \begin{cases} x = r(\theta - \sin \theta), \\ y = r(1 - \cos \theta) \end{cases}$$

are parametric equations of the curve traced by the point P. This curve is known as the ***cycloid.***

Example 3. Find the locus of a point P on a circle of radius a which rolls on the inside of a circle of radius $4a$.

Take the center of the fixed circle as the origin and let the x-axis pass through a point M where the moving point P touches the large circle. Let angle $MOB = \theta$ radians. Now we have arc $PB =$ arc $MB = 4a\theta$ and arc $PB = a \times$ angle PCB. Therefore

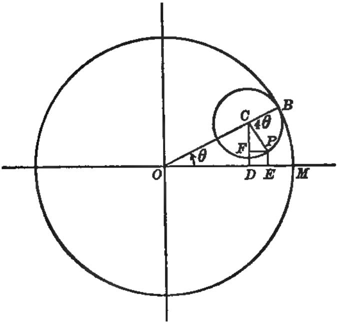


Fig. 236

$$a \times \text{angle } PCB = 4a\theta,$$

or
$$\text{angle } PCB = 4\theta.$$

But
$$\angle OCD + \angle DCP + \angle PCB = \pi.$$

Therefore
$$\frac{\pi}{2} - \theta + \angle DCP + 4\theta = \pi,$$

i.e.
$$\angle DCP = \frac{\pi}{2} - 3\theta.$$

Now if the point P is denoted by (x, y) we have

$$x = OE = OD + DE = OD + FP = OC\cos\theta + CP\sin\left(\frac{\pi}{2} - 3\theta\right)$$
$$= 3a\cos\theta + a\cos 3\theta = 4a\cos^3\theta,\text{*}$$
$$y = EP = DC - FC = OC\sin\theta - CP\cos\left(\frac{\pi}{2} - 3\theta\right)$$
$$= 3a\sin\theta - a\sin 3\theta = 4a\sin^3\theta;\text{*}$$

that is,

$$x = 4a\cos^3\theta,\quad y = 4a\sin^3\theta. \tag{12}$$

This curve is called the ***four-cusped hypocycloid.***

Example 4. PSP' is a double ordinate of an ellipse; Q is any point on the curve. If QP, QP' meet the x-axis in O and O', respectively, prove that $CO \cdot CO' = a^2$, where C is the center of the ellipse.

Let P be $(a\cos\theta_1, b\sin\theta_1)$, then P' is $(a\cos\theta_1, -b\sin\theta_1)$. Let Q be $(a\cos\theta, b\sin\theta)$. The equation of line PQ is

$$y - b\sin\theta = \frac{b(\sin\theta_1 - \sin\theta)}{a(\cos\theta_1 - \cos\theta)}(x - a\cos\theta)$$

---

* Prove that $\cos 3\theta = \cos(2\theta + \theta) = 4\cos^3\theta - 3\cos\theta,$
$\sin 3\theta = \sin(2\theta + \theta) = 3\sin\theta - 4\sin^3\theta.$

and its x intercept (*i.e.* CO) is

$$\frac{a(\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1)}{\sin\theta - \sin\theta_1}.$$

Similarly CO' is $$\frac{a(-\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1)}{\sin\theta + \sin\theta_1}.$$

The product $CO \cdot CO'$ gives a^2.

EXERCISES

1. A line of fixed length $2a$ moves with its ends always remaining on the coördinate axes. Find the locus of the mid-point of the line.

2. Find the locus of the middle points of chords of an ellipse drawn through the positive end of the minor axis.

3. Find the locus of a point P' on the radius CP of the cycloid (Fig. 235) if $CP' = b$ and $b < r$.

4. The same as Ex. 3, except $b > r$.

5. A circle of radius r rolls on the inside of a circle of radius a. Find the locus of a point P on the moving circle.

Ans. The hypocycloid $$\begin{cases} x = (a-r)\cos\theta + r\cos\dfrac{a-r}{r}\theta, \\ y = (a-r)\sin\theta - r\sin\dfrac{a-r}{r}\theta, \end{cases}$$

where θ is the same as in Example 3, § 254.

6. A circle of radius r rolls on the outside of a circle of radius a. Find the locus of a point P on the moving circle.

Ans. The epicycloid : $$\begin{cases} x = (a+r)\cos\theta - r\cos\dfrac{a+r}{r}\theta, \\ y = (a+r)\sin\theta - r\sin\dfrac{a+r}{r}\theta, \end{cases}$$

where θ is the same as in in Example 3, § 254.

7. The area of the triangle inscribed in an ellipse, if θ_1, θ_2, θ_3 are the eccentric angles of the vertices, is

$$\tfrac{1}{2}ab\,[\sin(\theta_2-\theta_3) + \sin(\theta_3-\theta_1) + \sin(\theta_1-\theta_2)]$$
$$= -2ab\sin\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}.$$

8. The coördinates of one extremity of a diameter of an ellipse are $(a\cos\theta_1, b\sin\theta_1)$. Show that the coördinates of one extremity of the conjugate diameter are $(-a\sin\theta_1, b\cos\theta_1)$.

PART IV. GENERAL ALGEBRAIC METHODS
THE GENERAL POLYNOMIAL FUNCTION

CHAPTER XVI

MISCELLANEOUS ALGEBRAIC METHODS

255. The Need of other Methods. We have hitherto considered special functions such as x^2, sin x, $\log_{10} x$, or special types of functions such as $mx + b$, $ax^2 + bx + c$, $\log_a x$, a^x; and we have studied their geometric and other applications. The study of more general types of functions, for example, the general polynomial of degree n,

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

of which $mx + b$, $ax^2 + bx + c$, $ax^3 + bx^2 + cx + d$ are special types, requires more powerful methods. Some of these we propose to consider in the present and the succeeding chapters.

256. Technique of Polynomials. We shall first recall the technique of the addition and multiplication of polynomials. We begin by noting that a polynomial

$$a_nx_n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

can be completely represented by so called *detached coefficients,* as follows:

$$a_n \;\; a_{n-1} \cdots a_1 \;\; a_0,$$

the place of each coefficient in this expression indicating the power of x to which it belongs. Thus, for example,

$$2 \quad -3 \quad 1 \quad 6$$

represents the polynomial $2x^3 - 3x^2 + x + 6$; and

$$1 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1$$

represents the polynominal $x^5 - 1$.*

To add two or more polynomials we need merely add the coefficients of like powers of x. Thus, the sum of $x^2 - 1$, $x^4 + 2x^3 + 4x^2 + 3x + 5$, and $2x^3 - 5x^2 + x + 1$ is given by

$$\begin{array}{rrrrrl}
 & & 1 & 0 & -1 & \\
1 & 2 & 4 & 3 & 5 & \\
 & 2 & -5 & 1 & 1 & \\
\hline
1 & 4 & 0 & 4 & 5 & = x^4 + 4x^3 + 4x + 5.
\end{array}$$

The analogy of this process with that of adding a column of numbers may be noted.

The product of two polynomials A and B is obtained by multiplying A by each term of B and adding the results. Why? The multiplication of two polynomials by the method of detached coefficients is also quite analogous to the familiar method of multiplying two integers. Thus the product of $2x^3 + 3x^2 - x - 2$ by $x^2 + x + 4$ is given by

$$\begin{array}{rrrrrrl}
2 & 3 & -1 & -2 \times 1 & 1 & 4 & \\
\hline
2 & 3 & -1 & -2 & & & \\
 & 2 & 3 & -1 & -2 & & \\
 & & 8 & 12 & -4 & -8 & \\
\hline
2 & 5 & 10 & 9 & -6 & -8 & = 2x^5 + 5x^4 + 10x^3 + 9x^2 - 6x - 8.
\end{array}$$

The student should convince himself, by multiplying these polynomials in the ordinary way, that the above method is indeed valid.

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* This method of representing polynomials will seem very natural, if we note the analogy with the familiar method of representing integers. The number 217 is simply a short way of writing $2 \times 10^2 + 1 \times 10 + 7$, *i.e.* the value of the polynomial $2x^2 + x + 7$, when $x = 10$. We have, therefore, been familiar with the method of detached coefficients from the time when we first learned to write numbers.

257. The Division Transformation. A polynomial B is said to be a ***factor*** of a polynomial A, if there exists a polynomial Q such that $A = BQ$. A is then said to be ***divisible*** by B. If no such polynomial Q exists, and if the degree of B is less than that of A, we may always determine polynomials Q and R, such that

$$A = BQ + R. \tag{1}$$

Furthermore, the *remainder* R can always be so determined that its degree is *less than the degree of B.*

The process whereby a given polynominal A is expressed in terms of another polynomial B in the form (1), *i.e.* the process of finding Q and R, when A and B are given, is called the ***division transformation.*** That it is always possible to find polynomials Q and R, if the degree of B is less than that of A, will be clear from the consideration of the following example.

Example. Given $A = 2x^4 + 5x^3 - 7x^2 + 12x - 5$ and $B = x^2 - 10x + 8$ to find a polynomial Q such that $A - BQ$ is a polynomial of degree less than that of B.

Since the term of highest degree in A is $2x^4$ and that in B is x^2, it appears that $A - 2x^2B$ can contain no term of degree higher than 3. In fact, we find $A - 2x^2B = 25x^3 - 23x^2 + 12x - 5$. Similarly, since the term of highest degree in $A - 2x^2B$ is $25x^3$, we see that the expression $(A - 2x^2B) - 25xB$ can contain no term of degree higher than 2. By continuing this process we shall arrive at a polynomial which is of degree less than that of B. The work may be arranged as follows.

$$\begin{array}{r|l}
A = 2x^4 + 5x^3 - 7x^2 + 12x - 5 & x^2 - 10x + 8 = B \\
B \cdot 2x^2 = 2x^4 - 20x^3 + 16x^2 \qquad\qquad & 2x^2 + 25x + 227 \\
\hline
A - B \cdot 2x^2 = \qquad 25x^3 - 23x^2 + 12x - 5 & \\
B \cdot 25x = \qquad 25x^3 - 250x^2 + 200x \quad\; & \\
\hline
A - B(2x^2 + 25x) = \qquad 227x^2 - 188x - 5 & \\
B \cdot 227 = \qquad 227x^2 - 2270x + 1816 & \\
\hline
A - B(2x^2 + 25x + 227) = \qquad 2082x - 1821 = R &
\end{array}$$

If the meaning of each step in the process is followed by means of the expressions written at the left,* it will be seen that the process has determined a polynomial

$$Q = 2x^2 + 25x + 227$$

such that

$$A - BQ = R,$$

where R is of degree less than that of B. The nature of the process shows that finally such a polynomial R will always be reached.

258. Remarks on the Division Transformation. While the process discussed in the last article is known as the *division transformation*, it is not a process of division. Only if we take a further step and divide both members of the identity (1) (§ 257) by B, to obtain

$$\frac{A}{B} = Q + \frac{R}{B}, \tag{2}$$

do we really divide A by B. The importance of this distinction lies in the fact that the relation (1) as derived above is valid without distinction for *all* values of the variable x involved. For in deriving the relation we made use only of the operations of multiplication and subtraction. However, the relation (2) becomes meaningless for all values of x for which $B = 0$.

We assumed in deriving the relation (1) that the degree of B was less than that of A. This is indeed necessary if Q is to be a proper polynomial. However, if the degree of B is equal to that of A, the same process will lead to a relation

$$A = B \cdot q + R, \tag{3}$$

where q is a constant. If the degree of B is greater than that of A, we may obviously write the trivial relation

$$A = B \cdot 0 + A \tag{4}$$

where A equals R and is by hypothesis of lower degree than B. If we consider a constant as a polynomial of degree 0, the last two cases may be included in the form (1). Our theorem then takes the general form

---

* These expressions are not of course a necessary part of the process. They are given here only to facilitate understanding.

Given any two polynomials A and B of degrees greater than 0, *then polynomials Q and R can always be found such that for all values of the variable we have $A = BQ + R$ where R is either zero or of degree less than that of B.*

Moreover, *the transformation of A to the form $BQ + R$ is unique, i.e.* there exist just one polynomial Q and one polynomial R satisfying the conditions of the theorem.

For, suppose there were a second pair, for example Q' and R'. We should then have $BQ + R = B'Q' + R'$, or $B(Q - Q') = R' - R$. But $R' - R$ is either zero or of degree less than that of B, while $B(Q - Q')$ is either zero or degree equal to or greater than that of B. Hence both are equal to zero and $R = R'$, $Q = Q'$.

EXERCISES

1. Add the following polynomials by means of detached coefficients:

(*a*) $2x^2 + 7x + 1$, $5x^2 + 2$, $3x^3 + 4x - 8$.
(*b*) $6t^2 + 5t + 1$, $9t^2 + 8t + 3$, $5t^3 + 2t + 1$.
(*c*) $ay^2 + by + c$, $2ay^2 + 3by + 4$, $3ay^2 + 5by + 7c$.
(*d*) $4a^2 + 3a + 2$, $6a^3 + 1$, $4a^2 + 2a + 3$, $2a^2 + 6a$.

2. Perform the following multiplications by means of detached coefficients:

(*a*) $x^3 + 2x^2 + x + 3$ by $2x + 1$.
(*b*) $x^3 + 3x^2 + 4$ by $x^3 + x + 2$.
(*c*) $a^4 + 1$ by $a^2 + 1$.
(*d*) $y^2 + 7y + 12$ by $y^3 + 3y + 2$.

3. In each of the cases below transform A into the form $BQ + R$, where R is of degree less than B. Also write down the corresponding form for A/B. Detached coefficients may be used to advantage.

(*a*) $A = 6x^4 + 7x^3 - 3x^2 - 24x - 20$, $B = 3x^2 + x - 6$.
(*b*) $A = 3x^4 + 2x^3 - 32x^2 - 66x - 35$, $B = x^2 - 2x - 7$.
(*c*) $A = 2x^5 + 5x^3 + 13x^2 + 2x$, $B = x^2 + 2x + 4$.
(*d*) $A = 4x^7 - 3x^5 - 19x^4 + 2x^3 - 4x^2 - 4x + 7$, $B = x^3 - x - 5$.

4. Determine m and n so that $x^4 - mx^3 + x^2 - nx + 1$ may be exactly divisible by $x^2 + 2x + 1$.

5. Prove the following propositions:

(*a*) If we multiply the dividend A by any constant as $k (k \neq 0)$, we multiply the quotient and the remainder by k.

(*b*) If we multiply the divisor by $k (k \neq 0)$, we divide the quotient by k but leave the remainder unchanged.

(*c*) If we multiply both dividend and divisor by $k (k \neq 0)$, we multiply the remainder by k but leave the quotient unchanged.

259. The Highest Common Factor of two Polynomials. Two polynomials A and B may or may not have a common factor of degree greater than 0, *i.e.* a polynomial F (of degree greater than 0) may or may not exist such that $A = FQ$, $B = FQ'$ where Q and Q' are also polynomials. If no such polynomial F of degree greater than 0 exists, then A and B are said to be *prime* to each other. If, on the other hand, they have a common factor of degree greater than 0, the one of the highest degree is called the *highest common factor* (H. C. F.).

THEOREM 1. *If A and B are polynomials with a common factor and M and N are any two polynomials, then any common factor of A and B is a factor of $AM + BN$.*

For let F be any common factor of A and B. Then we have $A = FQ$ and $B = FQ'$. Therefore

$$AM + BN = F(QM + Q'N),$$

which shows that F is a factor of $AM + BN$.

THEOREM 2. *If A, B, Q, R are polynomials such that $A = BQ + R$, the common factors of A and B are the same as the common factors of B and R.*

For, by Theorem 1, any common factor of B and R is a factor of A and hence a common factor of A and B. Moreover, from the relation $A - BQ = R$ and the last theorem, any common factor of A and B is a factor of B and R.

Successive applications of the division transformation therefore enable us to find the H. C. F. of two polynomials A and B as follows:

Using the division transformation on A and B, we may write

$$A = BQ + R,$$

where the degree of R is less than that of B. If $R = 0$, B is the H. C. F. If R is a constant different from zero, then A and B have no H. C. F. Why? If the degree of R is at least equal to 1, we may use the division transformation on B and R to obtain

$$B = RQ_1 + R_1,$$

where R_1 is of degree less than R. If $R_1 = 0$ then R is the H. C. F. of A and B. If R_1 is a constant different from zero, A and B are prime to each other. If R_1 is of degree at least equal to 1, proceed as before, expressing R in the form

$$R = R_1Q_2 + R_2.$$

This process may be continued until a remainder R_k is reached which is either zero, or a constant different from zero. If $R_k = 0$, then R_{k-1} is the H. C. F. sought. If R_k is a constant different from zero, then A and B are prime to each other.

Example 1. Find the H. C. F. of $4x^3 - 3x^2 + 7x - 1$ and $2x^2 - 3x + 1$.

$$\begin{array}{r|l}
A = 4x^3 - 3x^2 + 7x - 1 & 2x^2 - 3x + 1 = B \\ \cline{2-2}
4x^3 - 6x^2 + 2x \phantom{{}-1} & 2x + \frac{3}{2} = Q \\ \cline{1-1}
3x^2 + 5x - 1 & \\
3x^2 - \frac{9}{2}x + \frac{3}{2} & \\ \hline
\frac{19}{2}x - \frac{5}{2} = R &
\end{array}$$

Replace R by $x - \frac{5}{19} = R'$.

$$\begin{array}{r|l}
B = 2x^2 - 3x + 1 & x - \frac{5}{19} = R \\ \cline{2-2}
2x^2 - \frac{10}{19}x \phantom{{}+1} & 2x - \frac{47}{19} \\ \cline{1-1}
-\frac{47}{19}x + 1 & \\
-\frac{47}{19}x + \frac{235}{361} & \\ \hline
\frac{126}{361} = R_1 \neq 0 &
\end{array}$$

Therefore A and B are prime to each other.

Example 2. Find the H. C. F. of $x^3 - 2x^2 + x + 4$ and $3x^3 + 8x^2 + 3x - 2$.

The work may be arranged as follows. Since the H. C. F. of two polynomials is not altered in any essential way by multiplying or dividing either of them by any constant ($\neq 0$), we can avoid fractional coefficients.

$$
\begin{array}{r|c|l}
x^3 - 2x^2 + x + 4 & 3 & 3x^3 + 8x^2 + 3x - 2 \\
\underline{x^3 \qquad\quad - x \qquad} & & \underline{3x^3 - 6x^2 + 3x + 12} \\
-2x^2 + 2x + 4 & x - 2 & \quad 14x^2 - 14 \,\rfloor\, 14 \\
\underline{-2x^2 \qquad + 2} & & \qquad x^2 - 1 \\
2 \,\lfloor\, 2x + 2 & x - 1 & \qquad \underline{x^2 + x} \\
x + 1 & & \qquad -x - 1 \\
 & & \qquad \underline{-x - 1} \\
 & & \qquad\qquad 0
\end{array}
$$

Therefore the H. C. F. is $x + 1$.

EXERCISES

1. Find the H. C. F. of each of the following pairs of expressions:

(*a*) $x^3 + 2x^2 - 13x + 10$, $x^3 + x^2 - 10x + 8$.
(*b*) $3x^4 + 5x^2 + 2$, $x^6 - 4x^4 + 5x^2 - 2$.
(*c*) $x^3 - 2x^2 - 22x + 8$, $x^2 - 6x + 2$.
(*d*) $a^3 + 3a^2 - 3a - 5$, $a^3 - a^2 + 3a + 5$.
(*e*) $y^3 - y^2 - y + 1$, $y^2 + y + 1$.
(*f*) $b^4 + b^3 + 6b^2 + 5b + 5$, $b^5 + 4b^3 + b^2 - 5b + 1$.
(*g*) $1 + x - x^2 - 5x^3 + 4x^4$, $1 - 4x^3 + 3x^4$.
(*h*) $4x^4 - 5x^3 + x + 1$, $3x^4 - 4x^3 + 1$.
(*i*) $a^5 + a^3 + a + 1$, $a^2 + a + 1$.
(*j*) $x^5 - 1$, $x - 1$.

2. Prove that, if the coefficients of two polynomials are rational (or real), the coefficients of the H. C. F. are rational (or real).

3. If F is a factor of A but not of B, how does the H. C. F. of A and FB compare with the H. C. F. of A and B? In introducing and suppressing factors during the process of division, what precaution must be taken and why?

260. Functional Notation. We have already used special notations to represent special functions. Thus sin, cos, tan, log, etc. are special notations with which we are familiar. We shall now introduce a notation that is more general, for it is applicable to all kinds of functions. We shall use the symbols $f(x)$, $F(x)$, $\phi(x)$, $\cdots$ to represent various functions of x, and can then speak of the "f-function," "ϕ-function," etc., just as we speak of the sine function, logarithmic function, etc.

Moreover, such a notation can be used to represent the *value* of the function in question for a given value of the variable. Thus, if $f(x)$ is used to represent the function $x^2 + 2x - 1$, the symbol $f(2)$ denotes the value of this function when $x = 2$ (just as $\sin(\pi/2)$ means the value of $\sin x$, when $x = \pi/2$); *i.e.*

$$f(2) = 2^2 + 2 \cdot 2 - 1 = 7.$$

Similarly, with the meaning just given to $f(x)$, we should have

$$f(x+h) = (x+h)^2 + 2(x+h) - 1.$$

It should be noted that, when a certain function is called $f(x)$, then, throughout any discussion where this function is used, $f(x)$ always means that particular function and no other.

EXERCISES

1. Given $f(x) = 3x^2 + 2x - 4$, find $f(2)$, $f(h)$, $f(0)$, $f(x + 1/x)$.
2. Given $\phi(x) = x/(x-1)$, find $\phi(2)$, $\phi(x+h)$, $\phi(1-x)$, $\phi(10)$.
3. Given $F(x) = e^x + e^{-x}$, find $F(0)$, $F(1)$.
4. Given $f(x) = (x-1)/(x+1)$, prove that
$$\frac{f(y) - f(z)}{1 + f(y)f(z)} = \frac{y - z}{1 + yz}.$$
5. If $\phi(x+y) = \phi(x) + \phi(y)$, show that $\phi(3x) = 3\phi(x)$.
6. Given $f(x) = 2x\sqrt{1 - x^2}$, prove that
$$f(\sin x) = f(\cos x) = \sin 2x.$$
7. Given $f(x) = \dfrac{x - \dfrac{1}{x}}{x + \dfrac{1}{x}}$, find the value of $f\left(\sqrt{\dfrac{1+x}{1-x}}\right)$.
8. Given $\theta(x) = e^x + e^{-x}$, prove that
$$\theta(x+y)\theta(x-y) = \theta(2x) + \theta(2y).$$
9. Express the fact that the volume of a sphere is a function of its radius.

10. Given $F(x) = (x - 1/x)(x^2 - 1/x^2)(x^3 - 1/x^3)$, prove that

$$F(z) = -F(1/z).$$

11. Given that $f(n) = n\,!$,* prove that $(n+1)\,f(n) = f(n+1)$.

12. Given that $f(x) = x^2 + 2$, find $f[f(x)]$.

13. Given $f(x) = \sin x$, find $f(\pi/2)$, $f(\pi/3)$, $f(\pi)$.

14. Given $f(x) = \sin x$ and $\phi(x) = \cos x$, prove that

(a) $[f(x)]^2 + [\phi(x)]^2 = 1$.
(b) $f(x) \div \phi(x) = \tan x$.
(c) $f(x+y) = f(x)\,\phi(y) + f(y)\,\phi(x)$.
(d) $f(x) = \phi(\pi/2 - x)$.
(e) $f(-x) = -f(x)$.
(f) $\phi(x) = \phi(-x)$.

15. If $f(x) = \log_a x$, show that

(a) $f(x) + f(y) = f(xy)$.
(b) $f(x^n) = n f(x)$.
(c) $f(x/y) + f(y/x) = 0$.

16. What functions may $f(x)$ represent when

(a) $f(x+y) = f(x) f(y)$.
(b) $f(x+y) = f(x) + f(y)$.
(c) $f(x^n) = n\,f(x)$.
(d) $f\left(\dfrac{x}{y}\right) = f(y) - f(x)$.

261. The Remainder Theorem. *If a polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$.*

If $f(x)$ is the dividend, $x - a$ is the divisor, $Q(x)$ the quotient, and R the remainder, then

$$f(x) = (x - a)Q(x) + R \tag{5}$$

where R, since it is of lower degree than $x - a$, does not involve x at all, *i. e.* R has the same value for all values of x.

Since the values of the two members of this identity are equal to each other for all values of x, we have for the particular value $x = a$,

$$f(a) = R.$$

262. Factor Theorem. *If $f(x)$ is a polynomial and a is a number such that $f(a) = 0$, then $x - a$ is a factor of $f(x)$.*

The proof of this theorem is left as an exercise.

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* $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n$; that is, $2! = 2$, $3! = 6$, $4! = 24$, ...

EXERCISES

By use of the remainder theorem find the remainder when

1. $x^3 - 2x^2 + 3$ is divided by $x - 2$.

2. $x^{13} - 45x^{12} + 26x^5 + 12$ is divided by $x - 1$.

3. $x^{12} + 1$ is divided by $x+1$; by $x - 1$.

4. Show that -2 is a root of the equation $2x^3 + 3x^2 - 4x - 4 = 0$.

5. Show that $x^n + a^n$ is divisible by $x + a$ if n is odd.

6. Show that $x^n + a^n$ is not divisible by $x + a$ if n is even.

7. By means of the remainder theorem find k so that $x^3 + kx^2 + 2x + k$ may be exactly divisible by $x - 2$.

8. Find the polynomial in x of the second degree which vanishes when $x = 1$ and when $x = 2$, and which assumes the value 10 when $x = 3$.

263. Synthetic Division. We shall proceed to explain a simple method of performing the division transformation when the divisor has the form $x - k$, *i.e.* when the divisor is a binomial of the first degree in which the leading coefficient is 1.

Let the given polynomial be

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

and the divisor $x - k$.

The ordinary process of long division leads to

$$\begin{array}{l|l}
a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots+a_1x+a_0 & x-k \\ \cline{2-2}
\underline{a_nx^n-a_nkx^{n-1}\qquad\qquad\qquad\qquad\qquad\quad} & a_nx^{n-1}+(ka_n+a_{n-1})x^{n-2} \\
\qquad(ka_n+a_{n-1})x^{n-1} & \qquad\qquad\qquad\qquad\qquad +\cdots \\
\qquad\underline{(ka_n+a_{n-1})x^{n-1}-k(ka_n+a_{n-1})x^{n-2}} & \\
\qquad\qquad\qquad[k(ka_n+a_{n-1})+a_{n-2}]x^{n-2} &
\end{array}$$

It is now not difficult to see that

(*a*) the first coefficient in the quotient is a_n, *i.e.* the coefficient of the leading term in the dividend;

(*b*) the second coefficient in the quotient is obtained by multiplying the first coefficient of the quotient by k and adding to it the second coefficient of the dividend;

(*c*) the third coefficient of the quotient is obtained by multiplying the second coefficient of the quotient by k and adding to it the second coefficient of the dividend.

We may then arrange the work as follows:

$$\begin{array}{llll|l} a_n & a_{n-1} & a_{n-2} & a_{n-3}\cdots & \underline{k} \\ & ka_n & k(ka_n + a_{n-1}) & & \\ \hline a_n & ka_n + a_{n-1} & k(ka_n + a_{n-1}) + a_{n-2}\cdots & & \end{array}$$

Here the first line contains the coefficients of the dividend in order and the third line gives the coefficients of the quotient and the remainder in order. Every number in the third line, after the first, is obtained by multiplying the preceding number by k and adding to it the next number in the first line.

EXAMPLE 1. By synthetic division, find the quotient and the remainder when $x^4 - 2x^3 + x^2 + 3x - 2$ is divided by $x + 2$.

SOLUTION:

$$\begin{array}{rrrrr|l} 1 & -2 & 1 & 3 & -2 & \underline{-2} \\ & -2 & 8 & -18 & 30 & \\ \hline 1 & -4 & 9 & -15 & 28 & \end{array}$$

Hence the quotient is $x^3 - 4x^2 + 9x - 15$ and the remainder is 28.

EXAMPLE 2. If $f(x) = 2x^4 + 3x^3 + 7x^2 + 14x + 20$, find $f(-3)$.

$$\begin{array}{rrrr|rl} 2 & 3 & 7 & 14 & 20 & \underline{-3} \\ & -6 & 9 & -48 & 102 & \\ \hline 2 & -3 & 16 & -34 & \underline{122} & = f(-3).\ \textit{Ans.} \end{array}$$

EXERCISES

In the following exercises use synthetic division:

1. Find the remainder when $x^3 + 3x^2 - 6x + 2$ is divided by $x - 2$.

2. Find the remainder and the quotient when $x^4 - 3x^2 + 2x + 3$ is divided by $x + 3$.

3. Show that 3 is a root of the equation $x^3 - 4x^2 - 17x + 60 = 0$.

4. Find k so that 3 is a root of the equation $x^4 - 3x^2 + kx + 1 = 0$.

5. Is 5 a root of the equation $x^4 - x^2 + 7 = 0$?

264. Properties of Integers. We have already noticed (ftn., p. 403) that the familiar method of writing an integer greater than 9 represents it as the value of a polynomial. Integers and polynomials, therefore, have many properties in common. We may, for example, gain *an insight into the reason* for the rules of arithmetic used in adding a column of figures or in finding the product of two integers by comparing these rules with the technique of adding and multiplying polynomials discussed in § 256*. We shall proceed to discuss some of the properties of integers relating to divisibility, etc., which are valuable in our everyday use of numbers.

265. Prime and Composite Numbers. An integer greater than 1 that is divisible by no integer except itself and 1 is called a *prime number* or simply a *prime*. Thus 2, 3, 5, 7, 13 are prime numbers. Any integer (>1) not a prime is called a *composite* number. Any composite number is the product of two or more primes, thus $6 = 2 \cdot 3$, $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$. Any composite number n may be resolved into its prime factors in one and only one way. When resolved it has the form $n = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k}$. When a number has been resolved into its prime factors any question regarding its divisibility is readily answered by the following theorem.

A number a is divisible by a number b if and only if every prime factor of b occurs in a to at least as high a power as it occurs in b. This theorem follows readily from exercises 15 and 17 below. The proof is left to the student. As an illustration, if $a = 2 \cdot 3^3 \cdot 17^2 \cdot 37$ and $b = 2 \cdot 3^2 \cdot 17$ we recognize at once that a is divisible by b and that the quotient is $3 \cdot 17 \cdot 37$. If, on the other hand, b were $2^2 \cdot 3^3 \cdot 17$ then a would not be divisible by b.

The common factors of two integers are also readily found if the numbers have been resolved into their prime factors. Why? Two integers which have no common factor (>1) are said to be *prime to each other*.

The notion of prime numbers and the investigation of their properties is very ancient and to this day some of the most difficult problems of advanced mathematics relate to this field. Some of the properties are quite elementary, however, and have been listed below in exercises.

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* Carrying this comparison out in detail forms a valuable exercise. The familiar process of "carrying" a digit from one column to the next is about the only thing that requires special investigation.

EXERCISES

1. Prove that if a and b are both divisible by n, then $a+b$ and $a-b$ are divisible by n and $a \cdot b$ is divisible by n^2. Is a similar theorem true of polynomials?

2. Prove that a product of any number of integers is divisible by n if one of the integers is divisible by n. Is a similar theorem true of polynomials?

3. If $c = a \cdot b$ is divisible by n, must either a or b be divisible by n?

4. Prove that if a number a leaves a remainder r when divided by b, then the number obtained by adding to a any multiple of b will leave the same remainder.

5. Prove that if the last digit on the right of an integer is even, the integer is divisible by 2.

6. Prove that if the number formed by the last two digits of an integer is divisible by 4, then the number is divisible by 4.

7. Prove that if the number formed by the last three digits of an integer is divisible by 8, then the integer is divisible by 8.

8. Prove that if the last digit of an integer is 0 or 5 then the integer is divisible by 5.

9. Prove that if the sum of the digits of an integer is divisible by 3 (or 9) then the integer is divisible by 3 (or 9).

10. Prove that if the sum of the first, third, fifth, etc. digits of an integer is equal to the sum of the second, fourth, etc., the number is divisible by 11.

11. If the sum of the digits of an even number is divisible by 3, the number itself is divisible by 6.

12. Determine without performing the division whether the following numbers are divisible by 2, 3, 4, 5, 6, 8, 9, 11.

(*a*) 2562. (*c*) 123453. (*e*) 127898712. (*g*) 111111111111.
(*b*) 12345. (*d*) 2673. (*f*) 7325 × 8931. (*h*) 11111111112.

13. How would you recognize that a number is divisible by 45?

14. Prove that if the product $a \cdot b$ is divisible by a prime number p, either a is divisible by p or b is divisible by p. Is a similar theorem true for polynomials?

15. Prove that if a number c is a factor of ab and is prime to a, it must be a factor of b. Is a similar theorem true for polynomials?

16. Prove that the quotients of two numbers by their H. C. F. are two numbers prime to each other. Is a similar theorem true for polynomials?

17. Show that if a number is divisible by each of two numbers which are prime to each other, it is divisible by their product. Is a similar theorem true for polynomials ?

18. Show that the product of two numbers is equal to the product of their H. C. F. by their L. C. M. Is a similar theorem true of polynomials ?

19. Prove that the number of primes is unlimited.

[HINT. Suppose that the theorem were not true and that p were the greatest prime. Let $p_1, p_2, p_3, \dots, p_{n-1}$ be the set of all primes less than p and consider the number

$$p_1 p_2 p_3 \cdots p_{n-1} p + 1.$$

This number is not divisible by any of the primes $p_1, p_2, \cdots, p$. The rest of the proof should be obvious. This proof was first given by Euclid.]

20. By trying successive primes 2, 3, 5, 7, ⋯, determine whether or not 1009 and 1007 are primes. In this case we may stop with the prime 31. Why ? *Ans.* 1009 is prime.

21. Resolve into prime factors the numbers 604800 and 12259.

22. Is the number $2^6 3^{12} 5^3$ a perfect square ? Is it a perfect cube ?

23. Show that the relation $ab - cd = 1$, where a, b, c, d, represent integers, is not possible unless a and c are prime to each other.

24. Two consecutive integers are always prime to each other. Is this true also of any two numbers differing by 7 ?

25. What is the smallest cube of masonry that can be constructed of bricks 8 × 3 × 2 inches ? It is assumed that the bricks are placed so that any two equal sides are parallel.

266. Partial Fractions.

In certain problems it is sometimes found necessary to express a fraction in which the numerator and denominator are polynomials in one variable as the sum of several fractions each of which has a linear or at most a quadratic function in the denominator. In what follows it will always be assumed that the given fraction is a *proper fraction*, *i.e.* a fraction in which the degree of the numerator is less than that of the denominator. Any fraction which is not proper can be written as the sum of a polynomial and a proper fraction. Therefore our problem may be stated as follows : To express a proper fraction as the sum of several proper fractions.

The method of approach is to assume that the fraction can be expressed in the desired form and then seek to determine the numerators which in the assumed form are left undetermined. Four distinct cases arise.

CASE I. *When the denominator can be resolved into factors of the first degree all of which are real and distinct.*

EXAMPLE 1. Resolve into partial fractions

$$\frac{9x^2 - x - 2}{x^3 - x}.$$

The sum of the three fractions

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1},$$

will give a fraction whose denominator is $x^3 - x$. We therefore try to determine A, B, C so that

$$\frac{9x^2 - x - 2}{x^3 - x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}. \tag{6}$$

Clearing of fractions we have

$$9x^2 - x - 2 = A(x^2 - 1) + B(x^2 + x) + C(x^2 - x). \tag{7}$$

Since (7) is to be true for all values of x, we seek values of A, B, C, such that the coefficients of like powers of x will be equal, *i.e.* such that

$$A + B + C = 9, \quad B - C = -1, \quad -A = -2.$$

Solving these equations, we find $A = 2$, $B = 3$, $C = 4$. Hence

$$\frac{9x^2 - x - 2}{x^3 - x} = \frac{2}{x} + \frac{3}{x-1} + \frac{4}{x+1}.$$

CASE II. *When the denominator can be resolved into real linear factors some of which are repeated.*

EXAMPLE 2. Resolve into partial fractions

$$\frac{2x^2 - x + 2}{x(x-1)^2}.$$

The sum of the fractions

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2},$$

will give a fraction whose denominator is $x(x-1)^2$. Therefore we shall try to determine A, B, C so that

$$\frac{2x^2 - x + 2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing of fractions, we have

$$2x^2 - x + 2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Equating the coefficients of like powers of x, we have

$$A + B = 2, \quad -2A - B + C = -1, \quad A = 2.$$

Solving for A, B, C, we find $A=2$, $B=0$, $C=3$. Hence

$$\frac{2x^2-x+2}{x(x-1)^2}=\frac{2}{x}+\frac{3}{(x-1)^2}.$$

The assumptions to be made under Cases I and II are contained in the following rules.

(1) *For every unrepeated factor* $x-a$ *of the denominator, assume a fraction of the form* $A/(x-a)$.

(2) *For every repeated factor* $(x-a)^k$ *of the denominator, assume a sum of fractions of the form*

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_k}{(x-a)^k}.$$

Case III. *When the denominator contains quadratic factors which are not repeated.*

Example 3. Resolve into partial fractions

$$\frac{5x^2+4x+3}{(x+1)(x^2+1)}.$$

Let $$\frac{5x^2+4x+3}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}.$$

Clearing of fractions, we have

$$\begin{aligned}5x^2+4x+3&=A(x^2+1)+(Bx+C)(x+1)\\&=Ax^2+A+Bx^2+Bx+Cx+C.\end{aligned}$$

Equating coefficients,

$$A+B=5,\ B+C=4,\ A+C=3.$$

Therefore $A=2$, $B=3$, $C=1$. Hence

$$\frac{5x^2+4x+3}{(x+1)(x^2+1)}=\frac{2}{x+1}+\frac{3x+1}{x^2+1}.$$

Case IV. *When the denominator contains quadratic factors which are repeated.*

Example. Resolve into partial fractions

$$\frac{3x^4+x^3+8x^2+x+2}{x(x^2+1)^2}.$$

Let $$\frac{3x^4+x^3+8x^2+x+2}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

Then,

$$\begin{aligned}3x^4+x^3+8x^2+x+2&=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\\&=Ax^4+2Ax^2+A+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex.\end{aligned}$$

Equating coefficients we have

$$A + B = 3,\ C = 1,\ 2A + B + D = 8,\ C + E = 1,\ A = 2.$$

Hence, $A = 2,\ B = 1,\ C = 1,\ D = 3,\ E = 0.$

Therefore, $$\frac{3x^4 + x^3 + 8x^2 + x + 2}{x(x^2 + 1)^2} = \frac{2}{x} + \frac{x + 1}{x^2 + 1} + \frac{3x}{(x^2 + 1)^2}.$$

The assumptions to be made in Cases III and IV are contained in the following rules.

3. *Corresponding to every unrepeated factor of the form* $ax^2 + bx + c$, *assume the partial fraction* $$\frac{Ax + B}{ax^2 + bx + c}.$$

4. *Corresponding to every factor* $(ax^2 + bx + c)^k$, *assume the sum*
$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

EXERCISES

Resolve into partial fractions each of the following fractions.

1. $\dfrac{4x + 1}{(x - 1)(x + 1)(x + 3)}$.

2. $\dfrac{3x - 1}{x^2 - 4}$.

3. $\dfrac{2x + 1}{x^2(x - 4)}$.

4. $\dfrac{x^3 + 1}{x(x - 1)^3}$.

5. $\dfrac{1}{x^2(x + 1)}$.

6. $\dfrac{x^2 + 2x + 1}{x^3 + x}$.

7. $\dfrac{2x^2 - 1}{3x^3 + 3x}$.

8. $\dfrac{2x + 1}{x^3 + x^2 + x}$.

9. $\dfrac{1}{x(x^2 + 1)^2}$.

10. $\dfrac{3}{x^3 - 1}$.

11. $\dfrac{3x^2 - 5x + 4}{(x - 1)^3}$.

12. $\dfrac{x^2}{(x^2 - 1)^2}$.

13. $\dfrac{x^4}{(x^2 - 1)(x + 2)}$.

14. $\dfrac{x - 2}{(x + 1)x^2}$.

15. $\dfrac{2x^2 - x + 3}{x(x^2 - 1)(2x - 3)}$.

16. $\dfrac{1 + x^3}{(2 - x^2)(2 + x^2)}$.

17. $\dfrac{x^4 + x^2}{x^4 + x^2 + 1}$.

18. $\dfrac{3 - 2x^2}{(2 - 3x + x^2)^2}$.

19 $\dfrac{5x^3 + 2x + 1}{(x^2 + 1)(x - 1)^2}$.

20. $\dfrac{2x + 1}{x^2(x^2 + 1)^2}$.

CHAPTER XVII

PERMUTATIONS, COMBINATIONS, AND PROBABILITY THE BINOMIAL THEOREM

267. Definitions. Suppose that a group of n objects is given. Any set of r ($r \leqq n$) of these objects, considered without regard to order, is called a ***combination of the n objects taken r at a time.*** We often denote the objects in question, which may be of any kind, by letters, as $a, b, c, \cdots, k$. The number of combinations of these n letters taken r at a time is denoted by the symbol ${}_nC_r$. For example, the combinations two at a time of the four letters a, b, c, d are,

$$ab, ac, ad, bc, bd, cd.$$

Since there are 6 of these combinations in all, we have ${}_4C_2 = 6$.

On the other hand, any arrangement of r of these n objects in a definite order in a row is called a ***permutation*** of the n objects taken r at a time. The symbol ${}_nP_r$ is used to denote the number of such permutations.

For example, the permutations of the four letters a, b, c, d taken two at a time are

$$ab \quad ac \quad ad \quad bc \quad bd \quad cd \quad ba \quad ca \quad da \quad cb \quad db \quad dc.$$

Since there are 12 of these arrangements in all we have ${}_4P_2 = 12$. We have assumed in these examples that the objects are all different, and that the repetition of a letter within a permutation is not allowed.

268. Fundamental Principle. If a certain thing can be done in m different ways and if, when it has been done, a certain other thing can be done in p different ways, then both things can be done in the order stated in $m \times p$ different ways. For, corresponding to the first way of doing the first thing, there are p different ways of doing the second thing; corresponding to the second way of doing the first thing there are p different ways of doing the second thing; and so on for each of the m different ways of doing the first thing. Therefore there are $m \times p$ different ways of doing both things in the order stated. This fundamental principle may at once be extended to the following form.

If one thing can be done in m ways, and if, when it has been done, a second can be done in p ways, and if when that has been done, a third can be done in q ways, and so forth, then the number of ways in which they can all be done, taking them in the order stated, is $m \times p \times q \cdots$.

Example 1. There are five trails leading to the top of Mt. Moosilauke, N. H. In how many ways may I go to the top, and return by a different trail?

There are five ways I may go to the top and for each of these there are four ways I may descend. Therefore, the total number of ways in which I may make the round trip is 5 × 4 or 20.

Example 2. How many even numbers of two unlike digits can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9?

The digit in the units' place can be chosen in any one of 4 ways and the one in the tens' place can then be chosen in 8 ways. Therefore, 4×8 or 32 even numbers with two unlike digits can be formed from the given digits.

269. The Number of Permutations of *n* Different Things Taken *r* at a Time. The problem of finding the number of permutations of n different things taken r at a time can be stated as follows:

Find the number of ways in which we can fill r places when we have n different things at our disposal.

The first place can be filled in n ways, for we may take any one of the n things at our disposal. The second place can then be filled in $n-1$ ways, and hence the first and second places together can be filled in $n(n-1)$ ways. Why? When the first two places are filled, the third can be filled in $n-2$ ways. Reasoning as before, we have that the first three places can be filled in $n(n-1)(n-2)$ ways. Proceeding thus, we see that the number of ways in which r places can be filled is

$$n(n-1)(n-2)\cdots \text{ to } r \text{ factors},$$

and the rth factor is $n-(r-1)$ or $n-r+1$. Therefore the number of permutations of n different things taken r at a time is

$$(1) \qquad {}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

COROLLARY. If $r=n$, we have

$$(2) \qquad {}_nP_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!^*$$

EXAMPLE. Three students enter an office in which there are five vacant chairs. In how many ways can they be seated?

Here $n=5$, $r=3$. Hence ${}_5P_3 = 5\cdot 4\cdot 3 = 60$ ways.

270. The Permutations of n Things not all Different. *The number N of permutations of n things taken all at a time, of which p are alike, q others are alike, r others alike, and so on, is*

$$(3) \qquad N = \frac{n!}{p!\,q!\,r!\cdots}.$$

Suppose the n things are letters and that p of them are a, q of them b, r of them c, and so on.

Now, if in any of the N permutations we replace the p a's by p new letters, different from each other and also from the remaining $n-p$ letters, then by permuting these p letters among themselves without changing the position of any of the other letters we can form $p!$ new permutations. Therefore if this were done in each of the N permutations, we should obtain $N \cdot p!$ new permutations. In the same manner, if we replace the q b's by q new letters differing from each other and the remaining $n - q$ letters, the r c's by r new letters differing from each other and from the remaining $n - r$ letters, and so on, we then obtain $N \cdot p!\, q!\, r! \cdots$ new permutations. But the things are now all different and may be permuted in $n!$ ways. Therefore $N \cdot p! \cdot q! \cdot r! \cdots = n!$, or

---

* The product of all the integers from 1 to n is called ***factorial*** n, and is denoted by the symbol $n!$ or $\underline{n|}$. Thus $3! = 1\cdot 2\cdot 3 = 6$. A table of the values of $n!$ up to $n=10$ will be found at the end of the book.

$$N = \frac{n!}{p!\, q!\, r! \cdots}.$$

Example. How many different permutations of the letters of the word *Mississippi* can be formed taking the letters all together?

We have 11 letters of which 4 are s, 2 are p, 4 are i. Therefore the number of permutations is $11!/(4!\,4!\,2!) = 34650$.

EXERCISES

1. If there are six letter boxes, in how many ways can two letters be posted if they are not both posted in the same box? *Ans.* 30.

2. If there are six letter boxes, in how many ways can two letters be posted? *Ans.* 36.

3. Two dice are thrown on a table. In how many ways can they fall? *Ans.* 36.

4. Two coins are tossed on a table. In how many ways can they fall?

5. In how many ways can five coins fall on a table?

6. How many different permutations can be formed by taking five of the letters of the word *compare*?

7. Find the number of permutations that can be made from all the letters of the word (*a*) *assassination*; (*b*) *institutions*; (*c*) *examination*.

8. Given the digits 1, 2, 3, 4, 5, 6, 7, 8, 9. Find

(*a*) How many odd numbers of two digits each can be formed, repetition of digits being allowed.

(*b*) The same as (*a*), except that repetition of digits is not allowed.

9. How many even numbers less than 1000 can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, repetition of digits not being allowed?

10. In how many ways can a hand of ten cards be played one card at a time?

11. In how many ways can 3 different algebras and 4 different geometries be arranged on a shelf so that the algebras are together?

271. The Number of Combinations of n Different Things Taken r at a Time. *The number of combinations of n different things taken r at a time is*

$$(4) \qquad {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}.$$

For, each combination consists of a group of r different things which can be arranged among themselves in $r\,!$ ways. Therefore ${}_nC_r \cdot r\,!$ is equal to the number of permutations of n different things taken r at a time; that is, ${}_nC_r \cdot r\,! = {}_nP_r$, or

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}.$$

Corollary 1. The value of ${}_nC_r$ may be written in the form

$$(5) \qquad {}_nC_r = \frac{n\,!}{r\,!\,(n-r)\,!}.$$

This follows immediately from (4) if we multiply numerator and denominator by $(n-r)\,!$, since

$$n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)\,! = n\,!.$$

Corollary 2. The number of combinations of n different things taken r at a time is equal to the number of combinations of n different things taken $(n-r)$ at a time.

$${}_nC_{n-r} = \frac{n\,!}{(n-r)\,!\,(n-[n-r])\,!} = \frac{n\,!}{(n-r)\,!\,r\,!} = {}_nC_r.$$

The total number of ways in which a selection of some or all can be made from n different things is $2^n - 1$. For each thing may be disposed of in two ways, *i.e.* it may be taken or it may be left. Since there are n things, they may all be disposed of in 2^n ways. But among these 2^n ways is included the case in which all are rejected. Therefore the number of ways of making the selection is $2^n - 1$.

Example 1. In how many ways can a committee of 9 be chosen from 12 people?

The required number is

$$_{12}C_9 = {_{12}C_3} = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220.$$

Example 2. From 6 men and 5 women, how many committees of 8 each can be formed when the committee contains (1) exactly 3 women? (2) at least two women?

(1) The men may be chosen in $_6C_5$ ways, the women in $_5C_3$ ways. The number of ways in which both groups may be chosen together is $_6C_5 \cdot {_5C_3}$, or 60.

(2) Since each committee is to contain at least three women, it can be made up as follows:

(*a*) 5 men and 3 women.
(*b*) 4 men and 4 women.
(*c*) 3 men and 5 women.

Therefore the number of possible committees is

$$_6C_5 \times {_5C_3} + {_6C_4} \times {_5C_4} + {_6C_3} \times {_5C_5} = 155.$$

EXERCISES

1. Find $_{10}C_8$; $_{11}C_{10}$; $_{100}C_{99}$.

2. How many different committees of 6 men can be chosen from a group of 20 men?

3. There are 20 points in a plane, of which no three are in a straight line. How many triangles may be formed each of which has three of these points for its vertices?

4. How many planes may be determined by 25 points, no four of which are coplanar, if each of the planes is to contain three points?

5. How many different committees, each consisting of 5 republicans and 4 democrats, can be formed from 10 republicans and 8 democrats?

6. From 20 men how many groups of 11 men each can be picked? In how many of these groups will any given one of the 11 men be?

7. Out of 6 different consonants and 4 different vowels, how many linear arrangements of letters each containing 4 consonants and 3 vowels can be formed? *Ans.* $_6C_4 \times {_4C_3} \times 7!$.

8. From ten books, in how many ways can a selection of six be made,

(1) when a specified book is always included?

(2) when a specified book is always excluded?

272. Probability. If an event can happen in h ways and fail in f ways, and if each of these $f+h$ ways is equally likely, the (mathematical) *probability** of the event happening is

$$\frac{h}{h+f}$$

and the probability of its failing is $f/(h+f)$. An equivalent way of stating that $h/(h+f)$ is the probability of an event happening is to say that the *odds* are h to f in favor of the event or f to h against the event.

The probability of an event happening plus the probability of its failing is always equal to unity.

EXAMPLE 1. Suppose from a bag containing 3 red balls and 5 black ones, a ball is drawn at random, then the probability of its being red is $\frac{3}{8}$ and of its being black $\frac{5}{8}$. The chance that the ball is either red or black is $\frac{3}{8}+\frac{5}{8}=1$, or certainty.

EXAMPLE 2. From a bag containing 3 red balls and 5 black ones, two balls are drawn. Find the probability that (1) both are red, (2) both are black, (3) one is red and one is black.

Two balls can be drawn in ${}_8C_2$ or 28 ways. Two red balls can be drawn in ${}_3C_2$ or 3 ways. Therefore the probability of drawing two red balls is 3/28.

Two black balls can be drawn in ${}_5C_2$ or 10 ways. Therefore the probability of drawing two black balls is 10/28.

The number of ways of drawing one red ball and one black one is ${}_3C_1 \times {}_5C_1$, or 15. Therefore the probability of drawing a red and a black ball is 15/28.

EXAMPLE 3. Find the probability of throwing six with two dice. The total number of ways in which two dice can fall is 6×6 or 36. A throw of 6 can be made as follows: 1, 5; 5, 1; 4, 2; 2, 4; 3, 3; *i.e.* in 5 ways. Therefore the probability is 5/36.

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* The reason for the definition of mathematical probability may be made clear from the following considerations. Suppose a coin were tossed n times and fell heads h times and tails f times. If n is a finite number, h and f will in general not be equal. But as n is increased, $h/(h+f)$ and $f/(h+f)$ will approach nearer and nearer to 1/2, and thus we take 1/2 to be the probability of the coin falling heads

EXERCISES

1. In a single throw with one die, find the probability of throwing an ace.

2. In a single throw with two dice, find the probability of throwing a total of five; six; seven; eight.

3. In a single throw with two dice, find the probability of throwing at least five; six; seven; eight.

4. A bag contains 5 red balls, 6 green balls, 10 blue balls. Find the probability that, if 6 balls are drawn, they are (*a*) 2 red, 2 green, 2 blue; (*b*) 3 green, 3 blue; (*c*) 5 red, 1 green; (*d*) 6 blue.

5. Four coins are tossed. Find the probability that they fall two heads and two tails. *Ans.* 3/8.

6. In a throw with two dice, which sum is more likely to be thrown, 6 or 9?

7. Find the probability of throwing doublets in a throw with two dice.

8. Five cards are drawn from a pack of 52. Find the probability that

(*a*) there is one pair. [Two like denominations make a pair, for example, two aces.]

(*b*) Find the probability that there are three of a kind; (*c*) two pairs; (*d*) three of a kind and a pair; (*e*) four of a kind; (*f*) five cards of one suit.

9. Four cards are drawn from a pack of 52. Find the probability that they are one of each suit.

10. Seven boys stand in line. Find the probability that (*a*) a particular boy will stand at an end; (*b*) two particular boys will be together; (*c*) a particular boy will be in the middle.

11. A and B each throw two dice. If A throws 8, find the probability that B will throw a higher number.

12. Find the probability of throwing two 6's and one 5 in a single throw with three dice.

13. In tossing three coins find the probability that at least two will be heads.

14. If the probability that I shall win a certain event is $\frac{3}{4}$, what are the odds in my favor?

15. Find the probability of throwing an ace with a single throw of two dice. *Ans.* 11/36.

16. Which is more likely to happen, a throw of 4 with one die or a throw of 8 with two dice?

273. The Binomial Theorem for Positive Integral Exponents. Consider the product

$$(x+a)(x+a)\cdots(x+a) \qquad [\text{to } n \text{ factors}]$$

where n is any positive integer. One term of the product is x^n; it is obtained by taking the letter x from each parenthesis. There will be n terms $x^{n-1}a$, for the letter a may be chosen from any of the n parentheses which can be done in ${}_nC_1 = n$ ways. There will be ${}_nC_2$ terms $x^{n-2}a^2$, for the a's may be chosen from two of the n parentheses and the x from the remaining $n-2$ parentheses. In general, there will be ${}_nC_r$ terms $x^{n-r}a^r$, for the a's may be chosen from any r of the n parentheses, and the x's from the remaining $n-r$ parentheses. Therefore

$$(6) \quad (x+a)^n = x^n + {}_nC_1x^{n-1}a + {}_nC_2x^{n-2}a^2 + \cdots \\ + {}_nC_r x^{n-r}a^r + \cdots + a^n.$$

This formula for expanding $(x+a)^n$ is known as the ***binomial theorem.*** Since ${}_nC_r = {}_nC_{n-r}$, it follows that the coefficients of any two terms equidistant from the beginning and the end are equal. If we write $-a$ in place of a we have

$$(x-a)^n = x^{n-1} + {}_nC_1x^{n-1}(-a) + {}_nC_2x^{n-2}(-a)^2 + \cdots + (-a)^n,$$

or

$$(x-a)^n = x^n - {}_nC_1x^{n-1}a + {}_nC_2x^{n-2}a^2 - {}_nC_3x^{n-3}a^3 + \cdots + (-1)^n a^n.$$

EXAMPLE 1. Expand $(2x-y)^5$.

$$(2x-y)^5 = (2x)^5 - {}_5C_1(2x)^4y + {}_5C_2(2x)^3y^2 - {}_5C_3(2x)^2y^3 + {}_5C_4(2x)y^4 - y^5$$
$$= 32x^5 - 80x^4y + 80x^3y^3 - 40x^2y^3 + 10xy^4 - y^5.$$

EXAMPLE 2. Find the sixth term of $(2x-3y)^8$.

The sixth term is ${}_8C_5(2x)^3(-3y)^5$, or $-108{,}864\,x^3y^5$.

EXAMPLE 3. Find what term contains x^{11} in the expansion of $\left(x^2 - \frac{1}{x}\right)^{10}$.

Call it the t^{th} term. Then ${}_{10}C_{t-1}(x^2)^{11-t}\left(-\frac{1}{x}\right)^{t-1}$ is the term. In this term we want the exponent of x to be 11. Therefore $22 - 2t - t + 1 = 11$, or $t = 4$. The coefficient of this term is $-{}_{10}C_3 = -120$.

EXERCISES

Expand the following by the binomial theorem:

1. $(x-1)^5$.

2. $(2x+y)^6$.

3. $(2x-y)^7$.

4. $\left(1-\frac{1}{x}\right)^{10}$.

5. $\left(x-\frac{1}{x}\right)^8$.

6. $(z-xy)^5$.

7. $(0.9)^5$. [Hint. $0.9 = 1 - 0.1$.]

8. $(0.99)^3$.

Write down and simplify:

9. The 8th term of $(x-1)^{18}$.

10. The 5th term of $(2x-y)^{10}$.

11. The 7th term of $\left(\frac{4x}{5}-\frac{5}{4x}\right)^{10}$.

12. The 6th term of $(2x+3y)^{12}$.

13. The middle term of $(1-x)^{12}$.

14. The middle term of $(2x-y)^{14}$.

15. The middle terms of $(z-1/z)^{15}$.

Find the coefficient of

16. x^8 in the expansion of $\left(x^2-\frac{1}{x}\right)^{10}$.

17. x^{18} in the expansion of $\left(x^2+\frac{1}{x}\right)^{15}$.

18. x^{39} in the expansion of $\left(x^4+\frac{1}{x^3}\right)^{15}$.

19. x^{-17} in the expansion of $\left(x^4-\frac{a}{x^3}\right)^{15}$.

20. By considering the expansion of $(1+1)^n$, prove that

$$ {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1. $$

21. Prove $1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + \cdots + (-1)^n\, {}_nC_n = 0$.

MISCELLANEOUS EXERCISES

1. In how many ways can 10 boys stand in a row?

2. In how many ways can ten boys stand in a row when

(*a*) A given boy is always at a given end?
(*b*) A given boy is always at an end?
(*c*) Two given boys are always together?
(*d*) Two given boys are never together?

3. How many numbers of three digits each can be formed from the digits 1, 2, 3, 4, 5, 6, 7, when

(*a*) A repetition of digits is allowed?
(*b*) A repetition of digits is not allowed?

4. How many numbers of three digits each can be formed with the digits 2, 3, 5, 6, 7, 9, when

(*a*) The numbers are less than 500 and a repetition of digits is allowed?

(*b*) The numbers are greater than 500 and a repetition of digits is not allowed?

5. In how many ways can a consonant and a vowel be chosen from the letters of the word *vowels*?

6. Find n when

$$(a)\ {}_nC_2 = 45; \qquad (b)\ {}_nP_3 = 210; \qquad (c)\ {}_nC_2 = {}_nC_3.$$

7. Show that the number of ways in which n things can be arranged around a circle is $(n-1)!$.

8. In how many ways can 6 people sit around a round table?

9. How many signals can be made by hoisting 7 flags all at a time one above the other, if 2 are blue, 3 are white, and the rest are green?

10. How many different numbers of seven digits each can be formed with the digits 1, 2, 3, 4, 3, 2, 1, the second, fourth, and sixth digits being even?

11. How many handshakes may be exchanged among a party of 10 students if no two students shake hands with each other more than once?

12. A lodge has 50 members of whom 6 are physicians. In how many ways can a committee of 10 be chosen so as to contain at least 3 physicians?

13. A crew contains eight men; of these three can row only on the port side and two only on the starboard side. In how many ways can the crew be seated?

14. Find n when ${}_{n+2}C_4 = 11\,{}_nC_2$.

15. In how many ways can 18 books be divided into two groups of 6 and 12 respectively? *Ans.* ${}_{18}C_6$.

16. In how many ways can 12 students be divided into three groups of 4, 3, 5, respectively?

17. How many different amounts can be weighed with 1, 2, 4, 8, and 16 gram weights?

18. How many sums of money can be made with 5 one-cent pieces, 4 dimes, 2 half dollars, and 1 five-dollar bill?

19. In how many ways can four gentlemen and four ladies sit around a table so that no two gentlemen are adjacent? *Ans.* 144.

20. Prove $${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r.$$[*]

21. How many dominos are there in a set numbered from double blank to double six?

22. A railway signal has three arms and each arm can take three different positions. How many signals can be formed?

23. Prove ${}_{n+2}C_{r+1} = {}_nC_{r+1} + 2\,{}_nC_r + {}_nC_{r-1}$.

24. How many combinations of four letters each can be made from the letters of the word *proportion*? How many permutations?

Ans. 53; 758.

25. Find the probability that in a whist hand a player will hold the four aces.

26. Find the probability of drawing a face card from a pack of 52 playing cards.

27. If two tickets are drawn from a package of 15 marked 1, 2, ⋯, 15, what is the probability that they will both be marked with odd numbers? both with even numbers? both with numbers less than 10? both with numbers more than 10?

28. To decide on partners in a game of tennis four players toss their rackets. The 2 "smooths" and the 2 "roughs" are to be partners. What are the odds against the choice being made on the first throw?

29. Prove that the sum of the coefficients of the odd terms of a binomial expansion equals the sum of the coefficients of the even terms.

30. If n is an even integer, prove that there is a middle term in the expression of $(x + a)^n$ and that its coefficient is even.

31. Prove that ${}_nC_1 + 2\,{}_nC_2 + 3\,{}_nC_3 + \cdots n\,{}_nC_n = n(2)^{n-1}$.

32. Prove ${}_nC_1 - 2\,{}_nC_2 + 3\,{}_nC_3 + \cdots (-1)^{n-1} \cdot n \cdot {}_nC_n = 0$.

---

[*] An application of this formula is the construction of Pascal's Triangle. (${}_0C_0$ by definition will be assigned the value 1.)

$$\begin{array}{lllll}
{}_0C_0 & & & & \\
{}_1C_0 & {}_1C_1 & & & \\
{}_2C_0 & {}_2C_1 & {}_2C_2 & & \\
{}_3C_0 & {}_3C_1 & {}_3C_2 & {}_3C_3 & \\
{}_4C_0 & {}_4C_1 & {}_4C_2 & {}_4C_3 & {}_4C_4
\end{array}
\qquad\qquad
\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1
\end{array}$$

The formula in Ex. 20 shows that any number ${}_{n+1}C_r$ is equal to the number just above it, *i.e.* ${}_nC_r$, plus the number ${}_nC_{r-1}$ which is to the left of ${}_nC_r$. Thus for example ${}_4C_3 = {}_3C_3 + {}_3C_2$. We can, by means of this formula in Ex. 20, write down the next row. It is

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

The numbers in the nth row of the table are seen to be the coefficients of the terms in the expansion of $(x + a)^n$ (§ 273).

CHAPTER XVIII

COMPLEX NUMBERS

274. Definitions. We have already had occasions to refer to the so-called imaginary numbers. A number that arises as the result of extracting the square root or, indeed, any even root of a negative number is called an ***imaginary number.*** Thus $\sqrt{-2}$ is an imaginary number; the roots of the quadratic equation $x^2 + 8 = 0$, viz. $\pm 2\sqrt{-2}$, are imaginary numbers.

We have hitherto avoided the use of imaginary numbers as far as possible. It now becomes desirable to take them definitely into account, to learn how to work with them, and to gain some knowledge of their usefulness. Indeed, one of the primary objects of this chapter is to show that imaginary numbers have quite as concrete an interpretation as the real numbers, an interpretation which in many cases is of great service in the solution of concrete problems.

The letter i is used to represent the so-called ***imaginary unit;*** it is by definition such that $\boldsymbol{i^2 = -1}$.

Numbers of the form ib, where b is a real number different from zero, are called ***pure imaginary*** numbers.

Numbers of the form $a + ib$, where a and b are real numbers, are called ***complex*** numbers.

In the complex number $a + ib$, a is called the real part and ib the imaginary part. In a real number the imaginary part is zero; in a pure imaginary the real part is zero. A complex number $a + bi$ is imaginary if $b \neq 0$.

When two complex numbers differ only in the sign of the imaginary part they are said to be ***conjugate.*** Thus $3 + 2i$ and $3 - 2i$ are conjugate complex numbers.

275. Assumption. We *assume* that complex numbers obey the laws of algebra given in § 41. By applying this assumption we have symbolically for the sum and difference of the two complex numbers $a + ib$ and $c + id$,

$$a + ib \pm (c + id) = a \pm c + i(b \pm d).$$

That is, to add (subtract) complex numbers, add (subtract) the real and imaginary parts separately.

276. The Geometric Interpretation of the Imaginary Unit. We now seek a geometric interpretation of the imaginary unit i. To this end we recall the familiar representation of the real numbers as directed segments on a line, together with the interpretation of multiplication by -1 (§ 35). To multiply a real number a by -1 is equivalent geometrically to a rotation about the point O through two right angles of the segment OA which represents a (Fig. 237).

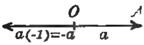

FIG. 237

Now, by definition, i is such a number that $i^2 = -1$. To multiply a real number a by -1 is then equivalent to multiplying it by i^2, *i.e.* by $i \cdot i$. Multiplying a real number a by i may, therefore, be interpreted geometrically as an operation which when performed twice is equivalent to a rotation about O in the plane through two right angles; *i.e.* to multiply a by i may be interpreted geometrically as equivalent to rotating OA about O in the plane through *one right angle.*

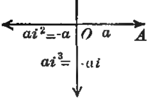

FIG. 238

The number ai will then be represented by a segment OB equal in length to OA whose direction makes with that of OA an angle of 90° (see Fig. 238). In the figure we have also indicated the result of multiplying a by $i^2 = i \cdot i = -1$ and by $i^3 = i \cdot i \cdot i = -i$.

Multiplying by $i^4 = i \cdot i \cdot i \cdot i = i^2 \cdot i^2 = 1$ is then to be interpreted as a rotation through four right angles.

EXERCISES

Give the conjugates of the following complex numbers:

1. $3 + 2i$. **2.** $3 - 4i$. **3.** $-5 - 3i$. **4.** $-8 + i$.

Simplify the following expressions:

5. $2(3 + 4i) - 4(1 - i)$. **7.** $\frac{4 - 3i}{2} - \frac{5 - 2i}{7}$.

6. $-4(1 - i) + 6(3 - 28i)$. **8.** $x + iy + ix + y$.

9. Prove that the sum of two conjugate complex numbers is a real number.

10. Is the following statement true? If the sum of two complex numbers is a real number, the complex numbers are conjugates. Explain.

11. Prove that every even power of i is equal to either 1 or -1.

12. Prove that every odd power of i is equal to either i or $-i$.

13. Find the value of $i + 2i^2 + 3i^3 + 4i^4$.

14. Find the value of $i^{28} + i^{45} + i^{63} + i^{39} + i^{44}$.

277. Vectors in the Plane. We have seen that, if any real number a is represented by a horizontal segment directed to the right or left according as the number a is positive or negative, then the imaginary number ai may be represented by a vertical segment directed upward or downward according as a is positive or negative. This suggests the possibility of representing other complex numbers by segments having other directions in the plane. Such a directed segment will represent a magnitude (the length of the segment) and a direction. Therefore such a segment can be used to represent a variety of concrete quantities that are not merely geometric; *e.g.* a force of a given magnitude and acting in a given direction; a velocity, meaning thereby the speed (magnitude) and the direction in which a body moves; etc. Such quantities having both direction and magnitude are called ***vectors***, and, if the directions are restricted to lie in the same plane, they are called ***plane vectors***. Any plane vector may, then, be represented by a directed line-segment in the plane.

Two vectors are said to be *equal* if and only if they have the same magnitude and the same direction. Hence, from any point in the plane as initial point, a vector can be drawn equal to any given vector in the plane.

278. Addition of Vectors. The addition of vectors in the plane proceeds according to a definition analogous to the geometric addition of directed line-segments discussed in § 35. If we are given two vectors AB and BC, we may conceive the first to represent a motion from A to B and the second a motion from B to C. The sum of the two vectors then represents, by definition, the net result of moving from A to B and then from B to C, *i.e.* the motion from A to C. The sum of the vectors AB and BC is then the vector AC (Fig. 239). In symbols

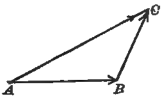

FIG. 239

$$AB + BC = AC.$$

In other words, the sum of two vectors is the vector from the initial point of the first to the terminal point of the second, when the vectors are so placed that the initial point of the second coincides with the terminal point of the first. From this definition it follows immediately that, if two vectors issue from the same point O, their sum is the diagonal, issuing from O, of the parallelogram of which the two given vectors form two adjacent sides (Fig. 240).*

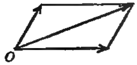

FIG. 240

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[*] If the vectors represent two forces, this shows that the sum of the vectors represents the resultant of the forces according to the law known as "the parallelogram of forces."

279. The Components of a Vector. The projection of a vector on a given line is called its *component* parallel to the line. Thus in Fig. 241 the directed segment M_1M_2 is the horizontal component of the vector AB, and the directed segment N_1N_2 is its vertical component. Moreover,

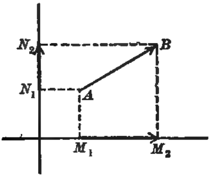

FIG. 241

$$\text{vector } AB = \text{vector } N_1N_2 + \text{vector } M_1M_2.$$

If the horizontal and the vertical components of a vector are known, then the vector is known. Why?

280. The Complex Number $x+iy$ and the Points in the Plane. Let OP (Fig. 242) be any vector issuing from O, and let the horizontal vectors issuing from O be represented by the positive and negative real numbers (and zero). We have seen that the numbers of the form ai can be represented by the vertical segments issuing from O. Here a is a real number and i is a vector of unit length. The horizontal component of OP will then be a certain real number x, and the vertical component a certain pure imaginary number iy. The vector OP will then be equal to the sum of these two components, *i.e.*

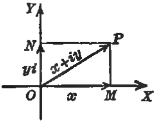

FIG. 242

$$OP = x + iy.$$

Conversely, every number of the form $x + iy$ represents a definite vector in the plane. If its initial point is at the origin of a system of rectangular coördinates (with equal units on the two axes), its terminal point is the point (x, y).

We have hitherto used vectors in the plane to represent the complex numbers. If we think of these vectors as all having their initial points at O, each vector determines uniquely, and is uniquely determined by, its terminal point. Hence, we can also use a complex number to represent a point in the plane, viz. the number $x + iy$ will represent the point whose rectangular coördinates are (x, y).

Example 1. Represent by means of vectors the complex numbers $2 + 2i$ and $1 + 6i$. Find the vector that represents their sum.

In Fig. 243 the vector OA represents the complex number $2 + 2i$, and the vector OB represents the complex number $1 + 6i$. The sum of these two complex numbers is represented by the vector OC. Why?

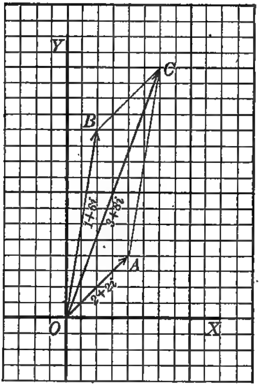

Fig. 243

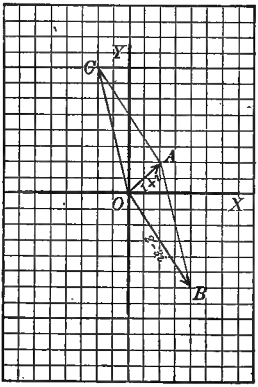

Fig. 244

Example 2. Find the vector that represents $(1 + i) - (2 - 3i)$.

To find this vector, find the vectors, OA and OB, that represent $1 + i$, and $2 - 3i$, and determine OC so that OA is the diagonal through O of the parallelogram of which OB and OC are adjacent sides (Fig. 244). Note that the vector OC is equal to the vector BA.

281. Equal Complex Numbers. *If* $x + iy = 0$, *then* $x = 0$ *and* $y = 0$. For, if $x + iy = 0$, and $y \neq 0$, we should have $x/y = -i$, which is impossible. Why?

If $x_1 + iy_1 = x_2 + iy_2$, *then* $x_1 = x_2$ *and* $y_1 = y_2$. For, by transposing terms, we have $(x_1 - x_2) + i(y_1 - y_2) = 0$. Hence, we have $x_1 = x_2$ and $y_1 = y_2$.

Thus, *two complex numbers are equal if and only if the real part of the first is equal to the real part of the second, and the imaginary part of the first is equal to the imaginary part of the second.* Geometrically, two complex numbers are equal if and only if they represent the same point.

282. The Polar Form of a Complex Number. Connect the point $P(x, y)$ (Fig. 245), which represents the complex number $x + iy$, to the origin O. If we let (ρ, θ) $(\rho \geqq 0)$ be the polar coördinates of P (O being the origin and OX the initial line), then for any position of the point P we have

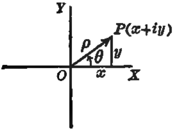

FIG. 245

$$(1) \qquad \begin{cases} x = \rho \cos \theta, \\ y = \rho \sin \theta. \end{cases}$$

Therefore, the complex number $x + iy$ may be written in the form

$$(2) \qquad x + iy = \rho(\cos \theta + i \sin \theta). \qquad (\rho \geqq 0.)$$

This form of complex number $x + iy$ is called the ***polar form.*** The angle θ is called the ***angle*** or the ***argument,*** and the length ρ is called the ***absolute value****[*] of the complex number.

EXAMPLE. Find the angle, the absolute value, and the polar form of the complex number $2 + i\,2\sqrt{3}$.

Plot the complex number (Fig. 246). Now we have $\rho = \sqrt{x^2 + y^2}$. Hence $\rho = \sqrt{4 + 12} = 4$. Moreover $\tan \theta = \sqrt{3}$, *i.e.* $\theta = 60°$. That is, the absolute value is 4 and the angle is 60°. Therefore the polar form is $4(\cos 60° + i \sin 60°)$.

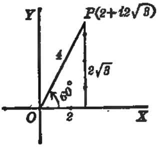

FIG. 246

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[*] Also sometimes called *modulus.*

EXERCISES

In the following exercises represent by vectors the numbers in parentheses, and their sum or difference as the case may be:

1. $(3+i)+(-4+2i)$.

2. $(1+3i)-(5-6i)$.

3. $7-(5+3i)$.

4. $(5-4i)-(-2-i)$.

5. $(3+2i)+(3+2i)$.

6. $(-4-4i)-6$.

Represent by a point each of the following complex numbers:

7. $3+5i$.

8. $3-3i$.

9. $6+i$.

10. $-5-3i$.

11. $-3+6i$.

12. $7+i\sqrt{2}$.

In the following exercises, represent by points the numbers in parentheses, and their sum or difference as the case may be:

13. $(3+i)+(-4+2i)$.

14. $(1+3i)-(5-6i)$.

15. $7-(5+3i)$.

16. $(5-4i)-(-2-i)$.

17. $(3+2i)+(3+2i)$.

18. $(3+3i)-5$.

Find real values of x and y satisfying the equations:

19. $2x-iy=4y-6-4i$.

20. $x+ixy=y+5+36i$.

21. $(3x+6y+2)i-3y-x=8$.

22. $ixy+x+y=5+4i$.

23. $x^2+y^2=25-(3x+4y-25)i$.

24. $ix+iy=4i+5x$.

Find the angle and the absolute value of each of the following complex numbers. Represent the numbers in polar form:

25. $1+i\sqrt{3}$.

26. $5+5i$.

27. $1-i$.

28. $\frac{1}{2}-\frac{i\sqrt{3}}{2}$.

29. $3i$.

30. -8.

31. $-8i$.

32. $12+5i$.

33. Can the complex number $x+iy$, where x and y are real numbers, equal 7?

34. Under what circumstances is the sum of two complex numbers a real number?

Change the following complex numbers from the polar form to the form $x+iy$:

35. $3(\cos 30^\circ + i\sin 30^\circ)$.

36. $4(\cos 135^\circ + i\sin 135^\circ)$.

37. $\cos 210^\circ + i\sin 210^\circ$.

38. $2\sqrt{2}(\cos 225^\circ + i\sin 225^\circ)$.

39. $4(\cos 90^\circ + i\sin 90^\circ)$.

40. $8(\cos 180^\circ + i\sin 180^\circ)$.

283. Multiplication of Complex Numbers. Our assumption in § 275 allows us to multiply two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ as follows:

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).\end{aligned}$$

If the two numbers are written in polar form, the multiplication may be performed as follows:

$$\begin{aligned}x_1 + iy_1 &= \rho_1(\cos\theta_1 + i\sin\theta_1), \\ x_2 + iy_2 &= \rho_2(\cos\theta_2 + i\sin\theta_2).\end{aligned}$$

By actual multiplication, we have

$$\begin{aligned}&(x_1 + iy_1)(x_2 + iy_2) \\ &\quad= \rho_1\rho_2[\cos\theta_1\cos\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - \sin\theta_1\sin\theta_2] \\ &\quad= \rho_1\rho_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].^*\end{aligned}$$

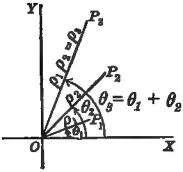

FIG. 247

Therefore, *the absolute value of the product of two complex numbers is equal to the product of their absolute values, and the angle of the product is equal to the sum of their angles.*

In Fig. 247 the points P_1 and P_2 represent the complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ respectively. The point P_3 represents $(x_1 + iy_1)(x_2 + iy_2)$.

284. Division of Complex Numbers. The quotient of two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ may be reduced to the form $a + ib$ if we make the denominator real by multiplying both numerator and denominator by the conjugate of the denominator. Thus,

$$\begin{aligned}\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} &= \frac{x_1x_2 + iy_1x_2 - ix_1y_2 - i^2y_1y_2}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{x_1y_2 - x_2y_1}{x_2^2 + y_2^2}.\end{aligned}$$

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* See § 138

If we write the two complex numbers in polar form and then perform the division, we have

$$\begin{aligned}\frac{\rho_1(\cos\theta_1+i\sin\theta_1)}{\rho_2(\cos\theta_2+i\sin\theta_2)}&=\frac{\rho_1(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{\rho_2(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}\\&=\frac{\rho_1[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]}{\rho_2(\cos^2\theta_2+\sin^2\theta_2)}\\&=\frac{\rho_1}{\rho_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)].\end{aligned}$$

Therefore, *the absolute value of the quotient of two complex numbers is equal to the quotient of their absolute values, and the angle of the quotient is equal to the difference of their angles.*

EXAMPLE 1. Find analytically and graphically the product $(1+i)(3+i\sqrt{3})$.

SOLUTION. *Analytically,*

$$(1+i)(3+\sqrt{3}\,i)=3+3\,i+\sqrt{3}\,i+\sqrt{3}\,i^2=(3-\sqrt{3})+i\,(3+\sqrt{3}).$$

Graphically, writing the complex numbers in polar form, we have

$$\sqrt{2}(\cos 45^\circ+i\sin 45^\circ)\text{ and }2\sqrt{3}(\cos 30^\circ+i\sin 30^\circ).$$

Therefore $\rho_1=\sqrt{2},\ \rho_2=2\sqrt{3},\ \theta_1=45^\circ,\ \theta_2=30^\circ.$

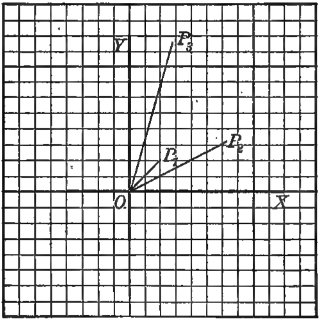

FIG. 248

Hence the absolute value of the product is $\rho_1\rho_2=2\sqrt{6}$ and the angle of product is 75°. In Fig. 248 the points P_1, P_2, and P represent respectively the complex numbers $1+i$, $3+i\sqrt{3}$, $(1+i)(3+i\sqrt{3})$.

Example 2. Find analytically and graphically the quotient

$$(3 + i\sqrt{3})/(1 + i).$$

Solution: *Analytically:*

$$\frac{3 + i\sqrt{3}}{1 + i} = \frac{3 + i\sqrt{3}}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{(3 + \sqrt{3}) - i(3 - \sqrt{3})}{2}.$$

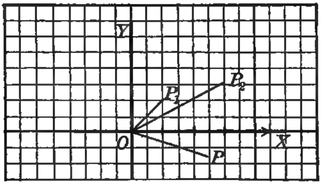

FIG. 249

Graphically, using the results in Ex. 1, we see in Fig. 249 that the points P_1, P_2, and P represent respectively the complex numbers $(1 + i)$, $(3 + i\sqrt{3})$, $(3 + i\sqrt{3})/(1 + i)$.

EXERCISES

Perform the following operations analytically and graphically:

1. $(1 + i)(2 + 2i)$.
2. $(1 + i\sqrt{3})(2 + i2\sqrt{3})$.
3. $(2i)(5)$.
4. $(1 + i)(-2 - 2i)(-1 + i\sqrt{3})$.
5. $\dfrac{3 + i\sqrt{3}}{1 + i}$.
6. $\dfrac{1 - i\sqrt{3}}{-3}$.
7. $\dfrac{5 + 5i}{1 - i}$.
8. $\dfrac{(1 - i)^2}{2 + i2\sqrt{3}}$.

Perform the following operations analytically:

9. $\dfrac{3 + i}{7 - i\sqrt{2}}$.
10. $\left(\dfrac{1 + i\sqrt{3}}{2}\right)^8$.
11. $\dfrac{3}{(2 + i)^2} + \dfrac{5}{(2 - i)^2}$.
12. $\dfrac{x + i\sqrt{1 - x^2}}{x - i\sqrt{1 - x^2}}$.
13. $\dfrac{8}{1 + i}$.
14. $(i^9 + i^{10} + i^{11} + i^{12})^7$.
15. $\dfrac{1 + 18i}{3 + 4i} - \dfrac{3 - 29i}{3 - 4i}$.
16. $\left(\dfrac{1 + i}{\sqrt{2}}\right)^4$.
17. $\dfrac{2 + 3i}{3 - 4i} + \dfrac{3 + 2i}{3 + 4i}$.
18. $\sqrt{7 + 24i}$.

285. DeMoivre's Theorem. The result of § 283 when applied to the product of any number of complex numbers leads to the following:

I. *The absolute value of the product of any number of complex numbers is equal to the product of their absolute values.*

II. *The angle of the product of any number of complex numbers is equal to the sum of their angles.*

If the above statements be applied to a positive integral power of a number, *i.e.* to the product of n equal factors, we obtain

$$[\rho(\cos\theta + i\sin\theta)]^n = \rho^n(\cos n\,\theta + i\sin n\,\theta). \tag{3}$$

For the special case $\rho = 1$ we obtain

$$\boldsymbol{(\cos\theta + i\sin\theta)^n = \cos n\,\theta + i\sin n\,\theta.} \tag{4}$$

This relation we have just proved for the case where n is a positive integer. It also holds when n is a *negative* integer. For we have

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}\\ &= \cos(-\theta) + i\sin(-\theta),\end{aligned}$$

and hence

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-p} &= [\cos(-\theta) + i\sin(-\theta)]^p\\ &= \cos(-p\,\theta) + i\sin(-p\,\theta).\end{aligned}$$

Further, if $n = 1/q$, where q is a positive or negative integer, we have, by what precedes,

$$(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \left[\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q\right]^{\frac{1}{q}} = \cos\frac{\theta}{q} + i\sin\frac{\theta}{q}, \tag{5}$$

and hence

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta. \tag{6}$$

This shows that relation (4) is valid for all *rational values of n*. It should be noted, of course, that relation (5) states merely that a certain q^{th} root of $\cos\theta + i\sin\theta$ is $\cos(\theta/q) + i\sin(\theta/q)$ and that a similar statement applies to relation (6). The fact expressed by (4) is known as ***De Moivre's theorem.****

286. Powers and Roots of Numbers.

De Moivre's theorem often enables us to compute an integral power of a complex number without difficulty, as the following example will show.

EXAMPLE 1. Find the value of $(2+2i)^5$. The polar form of this number is $2\sqrt{2}(\cos 45° + i\sin 45°)$. Hence

$$\begin{aligned}(2+2i)^5 &= (2\sqrt{2})^5(\cos 225° + i\sin 225°)\\ &= 128\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -128 - 128i.\end{aligned}$$

To find the nth roots of a number requires special methods.

EXAMPLE 2. Find the 5th roots of $2+2i$.

Here as in Ex. 1 we may write

$$2 + 2i = 2\sqrt{2}(\cos 45° + i\sin 45°)$$

and hence
$$(2+2i)^{\frac{1}{5}} = (2\sqrt{2})^{\frac{1}{5}}(\cos 9° + i\sin 9°).$$

But this is not the only number whose fifth power is $2+2i$. For we may write $2+2i = 2\sqrt{2}[\cos(45° + k\,360°) + i\sin(45° + k\,360°)]$, where k is any integer. That is to say,

$$(2+2i)^{\frac{1}{5}} = (2\sqrt{2})^{\frac{1}{5}}[\cos(9° + k\,72°) + i\sin(9° + k\,72°)].$$

For the values $k = 0, 1, 2, 3, 4$ we get the five numbers

$$(7)\quad \begin{cases} (2\sqrt{2})^{\frac{1}{5}}(\cos 9° + i\sin 9°), & (2\sqrt{2})^{\frac{1}{5}}(\cos 225° + i\sin 225°),\\ (2\sqrt{2})^{\frac{1}{5}}(\cos 81° + i\sin 81°), & (2\sqrt{2})^{\frac{1}{5}}(\cos 297° + i\sin 297°).\\ (2\sqrt{2})^{\frac{1}{5}}(\cos 153° + i\sin 153°), & \end{cases}$$

The succeeding values of k (*i.e.* $k = 5, 6, \cdots$) evidently give numbers equal to the preceding respectively. Each of the five numbers is a fifth root of $2+2i$; they are all different.

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* ABRAHAM DE MOIVRE (1667–1754), a mathematician of French descent who lived most of his life in England.

The general formulation of the problem of finding the nth root of a number $z = \rho(\cos\theta + i\sin\theta)$ is as follows. The most general form for z is

$$z = \rho[\cos(\theta + k\,360^\circ) + i\sin(\theta + k\,360^\circ)],$$

where k is an integer.

This gives, by De Moivre's theorem,

$$z^{\frac{1}{n}} = \rho^{\frac{1}{n}}\left[\cos\frac{\theta + k\,360^\circ}{n} + i\sin\frac{\theta + k\,360}{n}\right].$$

The n values $k = 0, 1, 2, \cdots, n-1$ give n different values for $z^{1/n}$ and no more values are possible. Why? Here $\rho^{1/n}$ means the numerical nth root of the positive number ρ. We have then: *Every complex number* ($\neq 0$) *has just n nth roots.* These n roots all have the same absolute value; their angles may be arranged in order in such a way that every two successive ones differ by $360^\circ/n$.

EXERCISES

By using De Moivre's theorem find the indicated powers, roots, and products.

1. $(4 + i\,4\sqrt{2})^6$.

2. $(\cos 10^\circ + i\sin 10^\circ)^9$.

3. $(1 + i)^{16}$.

4. $(3 + i\sqrt{3})^{10}$.

5. $(-1 - i\sqrt{3})^5$.

6. $(-2 + 2i)^4$.

7. $[3(\cos 15^\circ + i\sin 15^\circ)]^{15}$.

8. $[2(\cos 20^\circ + i\sin 20^\circ)][3(\cos 70^\circ + i\sin 70^\circ)]$.

9. $[2 + 2i][\sqrt{3} + i]$.

10. $(3 - 3i)(-1 + i\sqrt{3})$.

11. $\sqrt[3]{4 + i\,4\sqrt{2}}$.

12. $\sqrt{3 + i\sqrt{3}}$.

13. $\sqrt[6]{-4 + 4i}$.

14. $\sqrt[6]{8(\cos 60^\circ + i\sin 60^\circ)}$.

15. $\sqrt[3]{-1 - i\sqrt{3}}$.

16. $\sqrt[9]{\cos 45^\circ + i\sin 45^\circ}$.

17. $\sqrt[3]{27\,i}$.

18. The cube roots of 1.

19. $\sqrt[3]{-8}$.

20. $\sqrt[5]{32}$.

21. Prove that the n nth roots of a given number z are represented by the vertices of a regular polygon of n sides whose center is at the origin.

287. Applications in Trigonometry. De Moivre's theorem may be used to advantage in certain trigonometric problems.

I. *To express* $\cos n\theta$ *and* $\sin n\theta$ *in terms of* $\cos\theta$ *and* $\sin\theta$.

We have the relation

$$\begin{aligned}\cos n\theta + i\sin n\theta &= (\cos\theta + i\sin\theta)^n\\ &= \cos^n\theta + n\cdot i\cos^{n-1}\theta\sin\theta + \frac{n(n-1)}{2}i^2\cos^{n-2}\theta\sin^2\theta + \cdots\end{aligned}$$

If in this relation we equate the real and the imaginary parts we get the expressions desired.

Example 1. Express $\cos 6\theta$ and $\sin 6\theta$ in terms of $\cos\theta$ and $\sin\theta$.

The above method yields in this case:

$$\begin{aligned}\cos 6\theta + i\sin 6\theta &= (\cos\theta + i\sin\theta)^6\\ &= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta\\ &\quad + 6i\cos\theta\sin^5\theta - \sin^6\theta.\end{aligned}$$

Equating the real parts, we have

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

Equating the imaginary parts we get (after dividing by i)

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$$

II. *To express* $\cos^n\theta$ *and* $\sin^n\theta$ *in terms of sines and cosines of multiples of* θ. If we place $u = \cos\theta + i\sin\theta$, we have

$$u^k = \cos k\theta + i\sin k\theta, \qquad u^{-k} = \cos k\theta - i\sin k\theta.$$

Adding and subtracting these equations, we have

$$\left\{\begin{aligned} u^k + u^{-k} &= 2\cos k\theta,\\ u^k - u^{-k} &= 2i\sin k\theta,\end{aligned}\right. \tag{8}$$

for any integral value of k.

In particular when $k = 1$, we have

$$2\cos\theta = u + u^{-1}, \qquad 2i\sin\theta = u - u^{-1}.$$

It follows that

$$2^n\cos^n\theta = (u + u^{-1})^n = u^n + nu^{-2} + \frac{n(n-1)}{2}u^{n-4} + \cdots + na^{-(n-2)} + u^{-n}.$$

The fact that the coefficients in the binomial expansion are equal in pairs makes it always possible to group the terms as follows:

$$2^n \cos^n\theta = (u^n + u^{-n}) + n(u^{n-2} + u^{-(n-2)}) + \cdots.$$

But the terms in parentheses on the right are equal respectively to $2\cos n\theta$, $2\cos(n-2)\theta$, $\cdots$. The following examples will make the method clear.

Example 2. Express $\cos^4\theta$ in terms of cosines of multiples of θ.

We set

$$\begin{aligned} 2^4\cos^4\theta &= (u + u^{-1})^4 \\ &= u^4 + 4u^2 + 6 + 4u^{-2} + u^{-4} \\ &= (u^4 + u^{-4}) + 4(u^2 + u^{-2}) + 6 \\ &= 2\cos 4\theta + 4\cdot 2\cos 2\theta + 6. \end{aligned}$$

Dividing both members by 2^4 we obtain the desired result

$$\cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

Example 3. Express $\sin^5\theta$ in terms of multiples of the angle θ.

We set

$$2^5 i^5 \sin^5\theta = (u - u^{-1})^5$$

or

$$\begin{aligned} 32i\sin^5\theta &= u^5 - 5u^3 + 10u - 10u^{-1} + 5u^{-3} - u^{-5} \\ &= (u^5 - u^{-5}) - 5(u^3 - u^{-3}) + 10(u - u^{-1}) \\ &= 2i\sin 5\theta - 5\cdot 2i\sin 3\theta + 10\cdot 2i\sin\theta. \end{aligned}$$

Whence

$$\sin^5\theta = \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

EXERCISES

Express each of the following in terms of $\cos\theta$ and $\sin\theta$.

1. $\cos 2\theta$ and $\sin 2\theta$.

2. $\cos 3\theta$ and $\sin 3\theta$.

3. $\cos 4\theta$ and $\sin 4\theta$.

4. $\cos 5\theta$ and $\sin 5\theta$.

5. Show that $\tan 4\theta = \dfrac{4\tan\theta\,(1 - \tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$.

6. Find $\tan 5\theta$ in terms of $\tan\theta$.

Express each of the following in terms of sines and cosines of multiples of θ:

7. $\sin^3\theta$.

8. $\cos^3\theta$.

9. $\sin^4\theta$.

10. $\cos^5\theta$.

11. $\cos^6\theta$.

12. $\sin^6\theta$.

MISCELLANEOUS EXERCISES

Solve the following equations and illustrate the results graphically.

1. $x^3 - 1 = 0.$ **3.** $x^5 - 32 = 0.$ **5.** $x^8 - 1 = 0.$

2. $x^3 + 1 = 0.$ **4.** $x^6 - 1 = 0.$ **6.** $x^5 + 1 = 0.$

7. Prove that

$$\cos n\theta = \tfrac{1}{2}[\cos\theta + i\sin\theta]^n + \tfrac{1}{2}[\cos\theta - i\sin\theta]^n.$$

8. Prove that

$$i\sin n\theta = \tfrac{1}{2}[\cos\theta + i\sin\theta]^n - \tfrac{1}{2}[\cos\theta - i\sin\theta]^n.$$

9. Prove that

$$\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\tfrac{1}{2}n\pi - n\theta\right) + i\sin\left(\tfrac{1}{2}n\pi - n\theta\right).$$

10. Prove that the product of the n nth roots of 1 is 1, if n is odd, and -1 if n is even.

11. Prove that the sum of the n nth roots of any number is 0.

12. Complete the discussions in § 287 to derive the following formulas.

I. (*a*)
$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta + \cdots,$$

(*b*)
$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta\sin^5\theta + \cdots.$$

II. (*a*)
$$\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \cdots\right].$$

(*b*)
$$\sin^n\theta = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}}\left[\cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \cdots\right],$$

if n is even; but

$$\sin^n\theta = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}}\left[\sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{2!}\sin(n-4)\theta + \cdots\right],$$

if n is odd.

CHAPTER XIX

THE GENERAL POLYNOMIAL FUNCTION THE THEORY OF EQUATIONS

288. The General Polynomial Function of Degree *n*. The general polynomial of degree n,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \quad (a_n \neq 0),$$

has already been defined (§ 255). We have already discussed in some detail special cases when the degree of $f(x)$ is 1, 2, 3, (Chapters III, IV, V). For these cases we proved that the function is always continuous, and we learned how to find the slope of the graph of the function at any point. It is our present purpose to extend these results and methods to a function represented by a polynomial of any degree.

289. The Slope of the Graph of *f*(*x*). Continuity. To find the slope of the graph of the equation

$$(1) \qquad y = f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

at any point $P_1(x_1, y_1)$ of this graph, we first find the slope $\Delta y/\Delta x$ of the secant P_1Q (Fig. 250) joining the point P_1 to any other point $Q(x_1 + \Delta x, y_1 + \Delta y)$ on the graph. To this end we must first calculate the value of Δy in terms of x_1 and Δx. We have

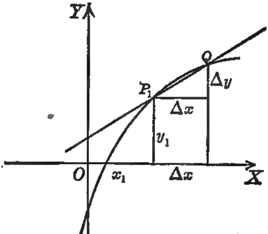

FIG. 250

$$
\begin{aligned}
y_1 + \Delta y = f(x_1 + \Delta x) \\
&= a_n(x_1 + \Delta x)^n + a_{n-1}(x_1 + \Delta x)^{n-1} + \cdots + a_1(x_1 + \Delta x) + a_0. \\
y_1 = f(x_1) &= a_n x_1^n + a_{n-1}x_1^{n-1} + \cdots + a_1 x_1 + a_0.
\end{aligned}
$$

By subtraction and proper grouping of terms we find

$$
\begin{aligned}
(2)\quad \Delta y &= f(x_1 + \Delta x) - f(x_1) \\
&= a_n[(x_1 + \Delta x)^n - x_1^n] + a_{n-1}[(x_1 + \Delta x)^{n-1} - x_1^{n-1}] \\
&\qquad + \cdots + a_1[(x_1 + \Delta x) - x_1].
\end{aligned}
$$

Each of the terms of this expression is of the form

$$
(3)\qquad a_k[(x_1 + \Delta x)^k - x_1^k],
$$

and the whole expression is equal to the sum of all terms obtained from (3) by letting k take on the values $k = n, n - 1, \cdots, 1$. Expanding the first term in the brackets, we obtain

$$
\begin{aligned}
a_k[(x_1 + \Delta x)^k - x_1^k] \\
&= a_k\left[x_1^k + kx_1^{k-1}\Delta x + \frac{k(k-1)}{2}x_1^{k-2}\Delta x^2 + \cdots + \Delta x^k - x_1^k\right] \\
&= a_k\left[kx_1^{k-1} + \frac{k(k-1)}{2}x_1^{k-2}\Delta x + \cdots + \Delta x^{k-1}\right]\Delta x.
\end{aligned}
$$

It is clear from this expression that for every value of k the expression (3) has Δx as a factor. Moreover the expression (2) for Δy is the sum of such terms as (3) for different values of k; and, since each of these terms has the factor Δx, their sum has the factor Δx. Hence, if we divide Δy by Δx, we have for the slope $\Delta y/\Delta x$ of P_1Q, the expression

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} = a_n\Big[nx_1^{n-1} &+ \frac{n(n-1)}{2}x_1^{n-2}\Delta x \\
&+ \text{terms with higher powers of } \Delta x\Big] && k = n. \\
&+ a_{n-1}\Big[(n-1)x_1^{n-2} + \frac{(n-1)(n-2)}{2}x_1^{n-3}\Delta x \\
&+ \text{terms with higher powers of } \Delta x\Big] && k = n-1. \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&+ a_2[2x_1 + \Delta x] && k = 2. \\
&+ a_1 && k = 1.
\end{aligned}
$$

The slope m of the graph is the limit approached by $\Delta y/\Delta x$ as Δx approaches zero (*i.e.* as Q approaches P_1 along the curve). This gives finally

$$(4) \qquad m = na_n x_1^{n-1} + (n-1)a_{n-1}x_1^{n-2} + \cdots + 2a_2x_1 + a_1$$

[Note that for the values $n = 3$ and $n = 2$ this reduces to the expressions previously derived for the cubic and quadratic functions.]

Moreover, it follows from the remark above, concerning the fact that Δx is a factor of Δy, that as Δx approaches zero, Δy approaches zero also. But this proves that $f(x)$ is continuous for every value of x. We have then the theorem:

Every polynomial $f(x)$ is a continuous function of x.

290. The Derived Function. In previous cases where we have considered the slope of a curve $y = f(x)$ we have always considered its value at some given point P_1 on the curve. As the point P_1 moves along the curve, however, the value of the slope in general changes. In other words, the slope itself may be considered as a function of x. This function is called the ***derived function*** or the ***derivative*** of $f(x)$. If the original function is denoted by $f(x)$, the derived function is denoted by $f'(x)$. In case of the polynomial $f(x)$ considered in the last article the derived function $f'(x)$ is obtained from the expression for the slope m by letting the given value x_1 become the variable x, *i.e.* if $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, we have the derived function

$$(5) \qquad f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

The derived function of any polynomial is readily written down from the following consideration. The derivative of any term a_kx^k is ka_kx^{k-1}; *i.e.* it is obtained *by multiplying the term by the exponent of x and reducing the exponent of x by* 1. Thus the derivative of x^3 is $3x^2$, of $10x^2$ is $20x$. The above expression for $f'(x)$ shows that the derivative of a polynomial is the sum of the derivatives of its terms. Thus the derivative of $5x^7 - 3x^4 + 7x^2 - 1$ may be written down at once; it is equal to $35x^6 - 12x^3 + 14x$. Observe that the derivative of a constant is 0.

The relation between the derived function $f(x)$ and the slope of the graph at any point, is expressed as follows:

The slope of the graph of the curve $y = f(x)$ at the point $x = x_1$ is equal to the value of the derived function for $x = x_1$, i.e. $m = f'(x_1)$.

Further, since the derived function of a polynomial is a polynomial, it follows from the theorem at the end of the last article, that the derived function of a polynomial $f(x)$ is a continuous function of x.

EXERCISES

Find $f'(x)$ when

1. $f(x) = x^3 + 4x^2 - 6x + 3$.
2. $f(x) = 5x^6 - 4x^3 + 6x^2 + 2x + 1$.
3. $f(x) = 7x^7 - 4x^3 + 2x + 19$.
4. $f(x) = 3x^5 - 4x^4 + 2x^3 + 3x^2 + 1$.
5. Find the equation of the tangent to $y = 4x^4 - 3x + 1$ at $(1, 2)$.
6. Find the equation of the tangent to $y = x^5 - 5x^2 + 2$ at the point $(1, -2)$.

291. The Graph of a Polynomial $f(x)$. In drawing the graph of a given polynomial of degree greater than 3, we may proceed as in the cases of polynomials of degrees 2 and 3. There are two general theorems to aid us:

(1) The graph of any polynomial is a continuous curve; in particular, the value of y does not become infinite except when x becomes infinite.

(2) The tangent to the graph at any point P turns continuously as P moves along the curve; *i.e.* the curve has no sharp corners and the tangent is nowhere vertical. (Why?)

We found in discussing the graphs of cubic functions that the values of x for which the slope is zero were particularly helpful, in view of the fact that they gave us, in general, the turning points (maxima and minima) of the graph. Let us apply these principles to an example.

Example. *Draw the graph of* $y = f(x) = \frac{1}{3}(3x^4 - 4x^3 - 12x^2 + 3)$.

We have $f'(x) = 4(x^3 - x^2 - 2x) = 4x(x-2)(x+1)$.

Hence $f'(x) = 0$ when $x = 0, 2, -1$.

We require next a table of corresponding values of x and y. Here synthetic division is often convenient. Thus, to find $f(x)$ when $x = 2$, we write

$$\begin{array}{rrrrrl} 3 & -4 & -12 & 0 & 3 & \underline{|2} \\ & 6 & 4 & -16 & -32 & \\ \hline 3 & 2 & -8 & -16 & | -29 = 3y. & \end{array}$$

Hence $y = -9\frac{2}{3}$ when $x = 2$.

When $x = 3$, we have

$$\begin{array}{rrrrrl} 3 & -4 & -12 & 0 & 3 & \underline{|3} \\ & 9 & 15 & 9 & 27 & \\ \hline 3 & 5 & 3 & 9 & 30 = 3y. & \end{array}$$

Hence $y = 10$ when $x = 3$.

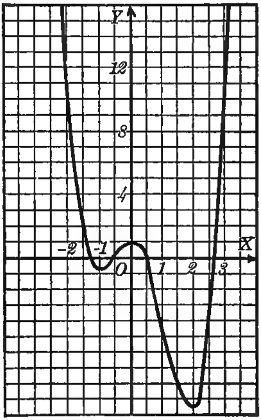

FIG. 251

We may note that since all the partial results 3, 5, 3, 9, 30 are positive, any value of $x > 3$ will give values of y greater than 10.

Finding the values of y for other values of x, we have the following table:

$x =$	-2	-1	0	1	2	3
$y =$	$11\frac{2}{3}$	$-\frac{2}{3}$	1	$-3\frac{1}{3}$	$-9\frac{2}{3}$	10
$m =$		0	0		0	

We have also indicated in the table the values of x for which m is zero. These data give us the graph exhibited in Fig. 251. This example suggests certain other general theorems regarding the graph of a polynomial, which are discussed in the following articles.

292. The Value of a Polynomial for Numerically Large Values of x. In the example of the last article we saw that for all values of $x>3$, the values of $f(x)$ were greater than 10; in fact, the nature of the synthetic division showed that as x increased indefinitely from $x=3$, the value of $f(x)$ increased indefinitely. Any polynomial $f(x)$ of degree n with real coefficients (§ 288) may be written in the form

$$f(x)=a_nx^n\left[1+\left(\frac{a_{n-1}x^{n-1}}{a_nx^n}+\frac{a_{n-2}x^{n-2}}{a_nx^n}+\cdots+\frac{a_0}{a_nx^n}\right)\right]$$
$$=a_nx^n\left[1+\left(c_{n-1}\frac{1}{x}+c_{n-2}\frac{1}{x^2}+\cdots+c_0\frac{1}{x^n}\right)\right].$$

Since the absolute value of a sum is equal to or less than the sum of the absolute values of its terms (§ 35), we have,

$$\left|c_{n-1}\frac{1}{x}+c_{n-2}\frac{1}{x^2}+\cdots+c_0\frac{1}{x^n}\right|\leqq\left|c_{n-1}\frac{1}{x}\right|+\left|c_{n-2}\frac{1}{x^2}\right|+\cdots+\left|c_0\frac{1}{x^n}\right|$$
$$<\left|\frac{1}{x}\right|(|c_{n-1}|+|c_{n-2}|+\cdots+|c_0|)<\frac{c}{|x|},\qquad(|x|>1).$$

where c is a positive number independent of x. Hence, if $|x|>c$, the value of the expression in square brackets above is certainly positive. Therefore for sufficiently large values $|x|$, the sign of $f(x)$ is the same as the sign of a_nx^n.

If a_n is positive and x becomes positively infinite, $f(x)$ is positive. If a_n is positive and x becomes negatively infinite, $f(x)$ is positive if n is even, and negative if n is odd. If a_n is negative and x becomes positively infinite, $f(x)$ is negative. If a_n is negative and x becomes negatively infinite, $f(x)$ is negative if n is even, and positive if n is odd.

As x increases indefinitely in absolute value, the value of $f(x)$ increases indefinitely in absolute value. For sufficiently large values of $|x|$, the sign of $f(x)$ is the same as the sign of a_nx^n.

In particular, this leads us to the following theorems.

If $f(x)$ is a polynomial of **even** *degree, the infinite branches of the graph of $y=f(x)$ are either both above the x-axis or both below the x-axis* (according as a_n is positive or negative).

If $f(x)$ is a polynomial of **odd** *degree, the infinite branches of the graph of $y=f(x)$ are on opposite sides of the x-axis* (below the x-axis on the left and above the x-axis on the right, if $a_n>0$; above the x-axis on the left and below on the right, if $a_n<0$).

From these theorems and from the continuity of the function $f(x)$ we derive the following corollary.

The graph of a polynomial $f(x)$ of odd degree with real coefficients must cross the x-axis at least once and, if it crosses more than once, it must cross it an odd number of times. The graph of a polynomial of even degree with real coefficients either does not cross the x-axis at all or it crosses it an even number of times.

293. The Zeros of a Polynomial $f(x)$. The Roots of the Equation $f(x)=0$. A value of x for which $f(x)=0$ is called ***a zero*** of $f(x)$; *i.e.* if $f(b)=0$, then b is a zero of $f(x)$. The zeros of $f(x)$ are, therefore, the values of x which satisfy the equation $f(x)=0$. The zeros of $f(x)$ are called the ***roots*** of the equation $f(x)=0$. The factor theorem (§ 261) tells us that if a is a zero of $f(x)$, then $x-a$ is a factor of $f(x)$. Since a polynomial of degree n cannot have more than n distinct factors of degree one, we may state the following theorem.

A polynomial $f(x)$ of degree n cannot have more than n distinct zeros.

Since at the turning points of $f(x)$ the slope is always zero, it follows from the fact that the derived function is of degree $n-1$ that *a polynomial $f(x)$ of degree n cannot have more than $n-1$ turning points (maxima and minima).*

294. The Number of Roots of $f(x) = 0$. We have seen that every quadratic equation has two roots which may be real or imaginary and which may be equal. We have also seen that every cubic equation $f(x) = 0$, whose coefficients are real, has at least one real root. If this root be r_1, we may write (§ 261), $f(x) = (x - r_1)Q(x)$, where $Q(x)$ is a polynomial of degree 2. The latter has two zeros, real or imaginary, so that any cubic function with real coefficients may be resolved into 3 linear factors,

$$f(x) = a_3(x - r_1)(x - r_2)(x - r_3).$$

It may be proved that *any polynomial (no matter whether the coefficients are real or imaginary) has at least one zero (real or imaginary).* This statement is called the ***fundamental theorem of algebra.*** We shall accept it as valid without proof, since its proof is too difficult for an elementary course.* From this theorem it is easy to prove the following theorem:

Any polynomial $f(x)$ of degree n may be resolved into n linear factors.

PROOF: By the fundamental theorem, $f(x)$ has one zero. Denote it by r_1. The factor theorem then gives

$$f(x) = (x - r_1)Q_1(x),$$

where Q_1 is a polynomial of degree $n - 1$. By the fundamental theorem, $Q_1(x)$ has a zero r_2. Hence

$$Q_1(x) = (x - r_2)Q_2(x), \text{ or } f(x) = (x - r_1)(x - r_2)Q_2(x).$$

Again, $Q_2(x)$ is a polynomial of degree $n - 2$. If $n > 2$, Q_2 has a zero, say r_3, which leads to the expression

$$f(x) = (x - r_1)(x - r_2)(x - r_3)Q_3(x),$$

where $Q_3(x)$ is a polynomial of degree $n - 3$. Continuing this process we find

---

* This theorem was first proved by GAUSS in 1797 (published 1799) when he was 18 years old. For proof see FINE, *College Algebra*, p. 588.

$$f(x) = (x - r_1)(x - r_2) \cdots (x - r_n)Q_n,$$

where Q_n is a constant which evidently must be a_n if $f(x)$ is $a_nx_n + \cdots + a_0$. We have then finally

$$f(x) = a_n(x - r_1)(x - r_2) \cdots (x - r_n).$$

Each of the numbers $r_1, r_2, \cdots, r_n$ is a root of the equation $f(x) = 0$. This proves the theorem just stated.

Moreover, no number different from $r_1, r_2, \cdots, r_n$ can be a root of this equation. For suppose s were such a number, then we should have $f(s) = a_n(s - r_1)(s - r_2) \cdots (s - r_n)$. Since each of these factors is under the hypothesis different from zero, the product $f(s)$ is different from zero. Some of the numbers $r_1, r_2, \cdots, r_n$ may be equal, however. This possibility leads to the following definitions. If $f(x)$ is exactly divisible by $x - r$ but not by $(x - r)^2$, then r is called a *simple* root of $f(x) = 0$. If $f(x)$ is exactly divisible by $(x - r)^2$ but not by $(x - r)^3$, then r is called a *double* root of $f(x) = 0$. If $f(x)$ is exactly divisible by $(x - r)^k$ but not by $(x - r)^{k+1}$ then r is called a *k-fold* root, or a root *of order k*. A root of order greater than one is called a *multiple* root. If $f(x)$ represents a polynomial, the equation $f(x) = 0$ is called an *algebraic* equation. Then we may state the last theorem as follows:

*Every algebraic equation of degree n has n roots and no more, if each root of order k is counted as k roots.**

EXERCISES

1. Is 1 a zero of the polynomial $x^7 - 3x^5 + 2x^4 - x + 3$?

2. Is 2 a zero of the polynomial $x^4 - 16$?

3. Is 3 a root of the equation $x^3 + 3x^2 + x - 3 = 0$?

---

* It is logically necessary to note the fact that, if exactly k of the roots $r_1, r_2, \cdots$ equal r, $f(x)$ is divisible by $(x - r)^k$ but not by $(x - r)^{k+1}$. Why?

4. Find k so that $x = 1$ is a root of the equation $x^3 + kx^2 - x + 1 = 0$.

5. Find k so that 2 is a root of the equation $x^3 + x^2 - kx + 3 = 0$.

6. How many roots has the equation $x^7 + x^3 + x + 3 = 0$? How many of these roots are positive?

7. How many roots has the equation $x^5 - 2x^4 + x^3 - 3x^2 + 2x - 1 = 0$? How many of these roots are negative?

8. Find graphically the real zeros of the functions
(a) $x^3 - x$. (b) $x^3 + 2x - 1$. (c) $x^3 + 3x + 2$. (d) $x^3 - x^2 - 6x + 8$.

Draw the graph of each of the following functions:

9. $y = \frac{1}{12}[3x^4 - 4x^3 - 24x^2 + 48x + 13]$.

10. $y = \frac{1}{12}[3x^4 + 8x^3 - 6x^2 - 24x - 12]$.

11. $y = \frac{1}{12}[3x^4 + 4x^3 - 12x^2 + 24]$.

12. $y = 2x^4 - 14x^3 + 29x^2 - 12x + 3$.

13. Prove, without assuming the fundamental theorem of algebra, that every algebraic equation of odd degree with real coefficients has at least one real root.

295. Successive Derivatives. The derived function of a polynomial $f(x)$ of degree n is a polynomial $f'(x)$ of degree $n - 1$. The derivative of $f'(x)$ is a polynomial of degree $n - 2$, is denoted by $f''(x)$, and is called the ***second derivative*** of $f(x)$. Similarly, the derivative of $f''(x)$ is called the ***third derivative*** of $f(x)$ and is denoted by $f'''(x)$. Similarly, the ***fourth, fifth,*** etc. ***derivatives*** may be found. The ***nth derivative*** of a polynomial of degree n is evidently a constant.

Thus, if $f(x) = x^4 - 3x^3 - 7x + 2$, we have $f'(x) = 4x^3 - 9x^2 - 7$, $f''(x) = 12x^2 - 18x$, $f'''(x) = 24x - 18$, $f^{iv}(x) = 24$.

296. Taylor's Theorem. The following formula is known as ***Taylor's theorem:***

$$(6)\quad f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^n(a)}{n!}(x - a)^n.$$

This formula enables us to express any polynomial in x as a polynomial in $x - a$, where a is any constant.

For example, if we have $f(x) = x^3 - 4x + 2$ and desire to express $f(x)$ in terms of $x + 1$, we first find $f'(x) = 3x^2 - 4$; $f''(x) = 6x$; $f'''(x) = 6$. The coefficients in the above formula are, for $a = -1$,

$$f(-1) = 5,\ f'(-1) = -1,\ f''(-1) = -6,\ f'''(-1) = 6.$$

Therefore we have, from (6),

$$x^3 - 3x + 4 = 5 - (x + 1) - 3(x + 1)^2 + (x + 1)^3.$$

Proof. We have seen in § 290 that the derivative of x^k is kx^{k-1}. Likewise the derivative of $(x - a)^k$ is $k(x - a)^{k-1}$. For if $y = (x - a)^k$, we have, as in § 289,

$$\begin{aligned} y + \Delta y &= (x + \Delta x - a)^k = [(x - a) + \Delta x]^k \\ &= (x - a)^k + k(x - a)^{k-1}\Delta x + \text{terms with a factor } \Delta x^2. \end{aligned}$$

Hence

$$\frac{\Delta y}{\Delta x} = k(x - a)^{k-1} + \text{terms with a factor } \Delta x,$$

and the limit of $\Delta y/\Delta x$ is obviously $k(x - a)^{k-1}$.

Let us now set

$$(7)\quad f(x) = A_0 + A_1(x - a) + A_2(x - a)^2 + \cdots + A_k(x - a)^k + \cdots.$$

We then have, by taking successive derivatives of both sides,

$$\begin{aligned} f'(x) &= A_1 + 2A_2(x - a) + \cdots + kA_k(x - a)^{k-1} + \cdots, \\ f''(x) &= 2A_2 + \cdots + k(k - 1)A_k(x - a)^{k-2} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f^{(k)}(x) &= k!\,A_k + \text{terms containing } (x - a) \text{ as a factor.} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

These relations must all be true for all values of x; hence they must hold when $x = a$. But this gives

$$f(a) = A_0,\ f'(a) = A_1,\ f''(a) = 2A_2, \cdots,\ f^k(a) = k!\,A_k, \cdots.$$

Hence

$$A_0 = f(a),\quad A_1 = f'(a),\quad A_2 = \frac{f''(a)}{2!}, \cdots,\quad A_k = \frac{f^k(a)}{k!}, \cdots.$$

By substituting these values in (7) above, we obtain Taylor's theorem as given in relation (6). Another form of Taylor's theorem is obtained by replacing x by $x+a$ in relation (6). This gives

$$(8)\quad f(x+a)=f(a)+f'(a)x+\frac{f''(a)}{2!}x^2+\cdots+\frac{f^n(a)}{n!}x^n.$$

EXERCISES

1. Write down the successive derivatives of the following polynomials:

(*a*) $x^3+4x^2-12x+17$.
(*b*) $2x^4-3x^3+8x^2-14x+18$.
(*c*) x^5+2x-1.
(*d*) $1-3x+4x^2+5x^3$.

2. Prove that the nth derivative of $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is equal to $a_nn!$.

3. Expand each of the following by Taylor's theorem:

(*a*) $x^3+4x^2-12x+17$ in terms of $x-1$.
(*b*) $2x^4-3x^3+8x^2-14x+8$ in terms of $x-2$.
(*c*) x^5+2x-1 in terms of $x+1$.
(*d*) $1-3x+4x^2+5x^3$ in terms of $x+2$.

4. By relation (8) in § 296 express each of the following as a polynomial in x:

(*a*) $f(x-1)$ if $f(x)=x^3+4x^2-12x+17$.
(*b*) $f(x-2)$ if $f(x)=2x^4-3x^3+8x^2-14x+8$.
(*c*) $f(x+1)$ if $f(x)=x^5+2x-1$.
(*d*) $f(x+2)$ if $f(x)=1-3x+4x^2+5x^3$.

297. Multiple Roots. If we apply Taylor's theorem successively to $f(x)$ and $f'(x)$, we obtain

$$(9)\quad f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots.$$

$$(10)\quad f'(x)=f'(a)+f''(a)(x-a)+\frac{f'''(a)}{2!}(x-a)^2+\frac{f^{\text{iv}}(a)}{3!}(x-a)^3+\cdots.$$

If $f(a) = 0$, the first relation shows that $x - a$ is a factor of $f(x)$; this constitutes a new proof of the factor theorem. If $f(x)$ is divisible by $x - a$ but not by $(x - a)^2$, it follows that $f(a) = 0$ and that $f'(a) \neq 0$. Hence by (10), or by the factor theorem, $f'(x)$ is not divisible by $x - a$. If $f(x)$ is divisible by $(x - a)^2$ but not by $(x - a)^3$, we have, from (9), $f(a) = 0$, $f'(a) = 0$, $f''(a) \neq 0$. We then conclude from (10) that if a is a double root of $f(x) = 0$, it is a simple root of $f'(x) = 0$. In general, if $f(x)$ is divisible by $(x - a)^k$ but not by $(x - a)^{k+1}$, relation (9) shows that $f(a) = f'(a) = f''(a) = \cdots = f^{k-1}(a) = 0$; $f^k(a) \neq 0$. Hence, by (10), $f'(x)$ is divisible by $(x - a)^{k-1}$ but not by $(x - a)^k$. This leads to the following theorem.

A simple root of $f(x) = 0$ is not a root of $f'(x) = 0$. A double root of $f(x) = 0$ is a simple root of $f'(x) = 0$. In general, a root of order k of $f(x) = 0$ is a root of order $k - 1$ of $f'(x) = 0$.

The following corollary of this theorem is evidently true.

Any multiple root of $f(x) = 0$ is also a root of $f'(x) = 0$. If $f(x)$ and $f'(x)$ have no common factor, $f(x) = 0$ has no multiple roots. If $\phi(x)$ is the H. C. F. of $f(x)$ and $f'(x)$, the roots of $\phi(x) = 0$ are the multiple roots of $f(x) = 0$.

Example 1. Examine for multiple roots the equation

$$f(x) = x^3 + x^2 - 10x + 8 = 0.$$

We have $f'(x) = 3x^2 + 2x - 10$. To find the H. C. F. of $f(x)$ and $f'(x)$ we proceed as in § 259:

$$\begin{array}{r|c|r}
3x^3 + 3x^2 - 30x + 24 & & 3x^2 + 2x - 10 \\
\underline{3x^3 + 2x^2 - 10x} & & \\
x^2 - 20x + 24 & x + 1 & \\
3x^2 - 60x + 72 & & 186x^2 + 124x - 620 \\
\underline{3x^2 + 2x - 10} & 3x & \underline{-186x^2 + 246x} \\
-62x + 82 & & 370x - 620
\end{array}$$

It is now clear that $f(x)$ and $f'(x)$ have no common factor. Hence we conclude that $f(x) = 0$ has no multiple roots.

Example 2. Examine for multiple roots the equation

$$f(x) = x^4 - 2x^3 + 2x - 1 = 0.$$

We have $f'(x) = 4x^3 - 6x^2 + 2$.

$$\begin{array}{r|c|r}
2x^4 - 4x^3 + 4x - 2 & x & 2x^3 - 3x^2 + 1 \\
2x^4 - 3x^3 + x \phantom{{}-2} & & 2x^3 - 4x^2 + 2x \\ \cline{1-1} \cline{3-3}
-x^3 + 3x - 2 & & x^2 - 2x + 1 \\
2x^3 - 6x + 4 & & x^2 - 2x + 1 \\ \cline{3-3}
2x^3 - 3x^2 + 1 & & 0 \\ \cline{1-1}
3x^2 - 6x + 3 & & \\
x^2 - 2x + 1 & 2x + 1 &
\end{array}$$

Hence $(x-1)^2$ is the H. C. F. of $f(x)$ and $f'(x)$, *i.e.* $x = 1$ is a triple root of $f(x) = 0$. The fourth root of $f(x) = 0$ is $x = -1$. How is it obtained?

EXERCISES

1. Examine for multiple roots each of the following equations:

(*a*) $x^3 - 3x^2 - 24x - 28 = 0$. (*b*) $x^5 + x^3 + 1 = 0$.
(*c*) $x^5 - 7x^3 - 2x^2 + 12x + 8 = 0$.
(*d*) $x^5 + x^4 - 9x^3 - 5x^2 + 16x + 12 = 0$.
(*e*) $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$.
(*f*) $x^6 - 3x^5 + 6x^3 - 3x^2 - 3x + 2 = 0$.

2. Prove that the graph of $y = f(x)$ is tangent to the x-axis at a point representing a multiple root.

3. Prove that the graph of $y = f(x)$ crosses or does not cross the x-axis at a point representing a multiple root according as the order of the root is odd or even. [Hint: Use Taylor's theorem.]

4. Prove that a root of order k of $f(x) = 0$ is a simple root of $f^{k-1}(x) = 0$.

298. Complex Roots. *If $a + bi$ (a, b real numbers, $i^2 = -1$) is a root of an algebraic equation $f(x) = 0$ with real coefficients, then $a - bi$ is also a root of the same equation.*

By hypothesis $a + bi$ is a root of the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0,$$

i.e. $f(a + bi) = a_n(a + bi)^n + a_{n-1}(a + bi)^{n-1} + \cdots + a_0 = 0.$

If each of the terms in the preceding expression be expanded by the binomial theorem, the powers of i reduced to their lowest terms ($i^2 = -1$, $i^3 = -i$, etc.), and terms collected, we obtain

$$f(a + bi) = P + Qi,$$

where P represents the sum of the terms independent of i and Q is the coefficient of i.

But since $P + Qi = 0$ by hypothesis, it follows from § 281, that both $P = 0$ and $Q = 0$. We wish to prove that $a - bi$ is a root of $f(x) = 0$; *i.e.* $f(a - bi) = 0$. To prove this we merely have to notice that $f(a - bi)$ may be obtained from the expression for $f(a + bi)$ by replacing i by $-i$. Therefore

$$f(a - bi) = P - Qi,$$

where P and Q represent the same quantities as before. But we have just shown that $P = 0$ and $Q = 0$. Therefore $f(a - bi) = 0$ or $a - bi$ is a root of $f(x) = 0$.

EXERCISES

1. Solve $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + i$.

2. Solve $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$, one root being i.

3. Solve $x^4 - 2x^3 + 5x^2 - 2x + 4 = 0$, one root being $1 - i\sqrt{3}$.

4. If $a + \sqrt{b}$ (a and b rational but $\sqrt{b}$ irrational) is a root of $f(x) = 0$ with rational coefficients, $a - \sqrt{b}$ is also a root.

[HINT: Show that $f(a + \sqrt{b})$ reduces to the form $P + Q\sqrt{b}$ where P and Q contain only integral powers of b and Q is the coefficient of $\sqrt{b}$. Since $P + Q\sqrt{b} = 0$, $P = 0$ and $Q = 0$. Why?]

5. Solve $2x^4 - 3x^3 - 16x^2 - 3x + 2 = 0$, one root being $2 + \sqrt{3}$.

6. Form an equation with rational coefficients, of which two of the roots are i and $1 + \sqrt{2}$.

7. Solve the equation $x^3 - (4 + \sqrt{3})x^2 + (5 + 4\sqrt{3})x - 5\sqrt{3} = 0$, if one root is $2 - i$.

8. Solve the equation $x^3 - (5 + i)x^2 + (9 + 4i)x - 5 - 5i = 0$ if one root is $1 + i$. Is $1 - i$ a root?

299. To Multiply the Roots of an Equation. *To transform a given equation $f(x) = 0$ into another whose roots are those of $f(x) = 0$ each multiplied by some constant k, multiply the second term of $f(x)$ by k, the third term by k^2, and so on, taking account of the missing terms if there are any.*

The required equation is $f(y/k) = 0$. For, if $f(x)$ vanishes when $x = a$, $f(y/k)$ will vanish when $y = ka$. Hence, if the given equation is $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, the required equation is

$$a_n\left(\frac{y}{k}\right)^n + a_{n-1}\left(\frac{y}{k}\right)^{n-1} + \cdots + a_0 = 0,$$

which on multiplication by k^n becomes

$$a_n y^n + ka_{n-1}y^{n-1} + k^2 a_{n-2}y^{n-2} + \cdots + k^n a_0 = 0.$$

If $k = -1$, we have, *the roots of $f(-x) = 0$ are equal respectively to those of $f(x) = 0$ with their signs changed.*

Example 1. Transform $x^3 - 4x^2 + 5 = 0$ into an equation whose roots are twice those of the given equation. The desired equation is $x^3 - 4(2)x^2 + 5(2)^3 = 0$, or $x^3 - 8x^2 + 40 = 0$.

Example 2. Transform $x^7 - 3x^5 + 4x^4 - 2x + 1 = 0$ into an equation whose roots are those of this equation with their signs changed. The result is $(-x)^7 - 3(-x)^5 + 4(-x)^4 - 2(-x) + 1 = 0$, or

$$x^7 - 3x^5 - 4x^4 - 2x - 1 = 0.$$

EXERCISES

Obtain equations whose roots are equal to the roots of the following equations multiplied by the numbers opposite.

1. $x^5 - 2x^3 + x + 1 = 0.$ (2)
2. $x^7 - 5x^3 + 2x - 1 = 0.$ (− 2)
3. $x^4 - x^2 + x + 1 = 0.$ (− 3)
4. $x^5 + x^4 - x^3 + x - 1 = 0.$ (2)

Obtain equations whose roots are equal to the roots of the following equations with their signs changed.

5. $x^7 - 6x^5 + 2x^4 - x + 1 = 0.$
6. $x^{15} - 1 = 0.$
7. $x^7 - x^6 + x^5 - x^4 - 2 = 0.$
8. $1 - x - x^2 - x^3 - x^4 - x^5 = 0.$

300. Variations in Sign. A *variation of sign* or *change of sign* is said to occur in $f(x)$ whenever a term follows one of opposite sign. Thus the equation $x^3 - 3x^2 + 7 = 0$ has two variations of sign.

If $f(x)$ has real coefficients and is exactly divisible by $x - k$, where k is positive, then the number of variations of sign in the quotient $Q(x)$ is at least one less than the number of variations of sign in $f(x)$.

Before proving this statement let us consider the process of dividing $f(x) = x^6 + x^5 - 3x^4 - 2x^3 - x^2 + 5x - 1$ by $x - 1$ and $f(x) = x^4 - x^3 + 4x^2 - 13x + 2$ by $x - 2$, making use of synthetic division.

$$\begin{array}{rrrrrrrr|l}
 & 1 & 1 & -3 & -2 & -1 & 5 & -1 & \underline{1} \\
 & & 1 & 2 & -1 & -3 & -4 & 1 & \\
\hline
Q(x) = & 1 & 2 & -1 & -3 & -4 & 1 & 0 &
\end{array}$$

$$\begin{array}{rrrrrr|l}
 & 1 & -1 & 4 & -13 & 2 & \underline{2} \\
 & & 2 & 2 & 12 & -2 & \\
\hline
Q(x) = & 1 & 1 & 6 & -1 & 0 &
\end{array}$$

It will be noted in these examples that $Q(x)$ has no variations except such as occur in the corresponding or earlier terms of $f(x)$ and that since $f(x)$ is exactly divisible by the given divisor, the sign of the last term of $Q(x)$ is opposite to that in $f(x)$. Let us now prove the statement in general.

Proof: From the nature of synthetic division it follows that the coefficients in $Q(x)$ must be positive at least until the first negative coefficient of $f(x)$ is reached. Then, or perhaps not until later, does a coefficient of $Q(x)$ become negative or zero, and then they continue negative at least until a positive coefficient in $f(x)$ is reached. Therefore $Q(x)$ has no variations except such as occur in the corresponding or earlier terms of $f(x)$. But by hypothesis $f(x)$ is exactly divisible by $x-k$ and hence the sign of the last term in $Q(x)$ must be opposite to that in $f(x)$. Therefore the number of variations of sign in $Q(x)$ must be *at least* one less than the number of variations of sign in $f(x)$.

301. Descartes's Rule of Signs. *The equation $f(x)=0$ with real coefficients can have no more positive roots than there are variations of sign in $f(x)$ and can have no more negative roots than there are variations of sign in $f(-x)$.*

PROOF: Let $r_1, r_2, \cdots, r_p (p \leqq n)$ denote the positive roots of $f(x)=0$. If we divide $f(x)$ by $x-r_1$, the quotient by $x-r_2$, and so on until the final quotient $Q(x)$ is obtained, then we know from the last theorem that $Q(x)$ contains at least p fewer variations of sign than $f(x)$. But the least number of variations of sign that $Q(x)$ can have is zero. Therefore $f(x)$ must have at least p variations, *i.e.* at least as many variations as $f(x)=0$ has positive roots.

Second, by § 299, we know that the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$ and, hence, by the first part of this proof, we know that their number cannot exceed the number of variations of sign in $f(-x)$.

It is important to notice that Descartes's rule of signs does not tell us how many positive and how many negative roots an equation has. It merely tells us that an equation *cannot have more than* a certain number of positive roots, and *cannot have more than* a certain number of negative roots.

EXAMPLE. What conclusions regarding the roots of the equation $x^7-4x^5+3x^2-2=0$ can be drawn from Descartes's rule?

The signs of $f(x)$ are $+-+-$, *i.e.* there are 3 variations and hence the equation has no more than 3 positive roots.

The signs of $f(-x)$ are $-++-$, *i.e.* there are two variations and hence the equation has no more than 2 negative roots.

But the equation is of degree 7 and has 7 roots. Therefore the equation has at least two imaginary roots. Can there be more than two imaginary roots?

EXERCISES

What conclusions regarding the roots of the following equations can be drawn from Descartes's rule?

1. $x^7 - 2x^6 + x^4 - 1 = 0.$

2. $x^5 + x^4 - x^3 + 1 = 0.$

3. $x^{28} - 34x^{12} + x - 45 = 0.$

4. $x^5 - 2x^4 + x^3 - x^2 - x + 1 = 0.$

5. $x^n - 1 = 0.$ (n odd)

6. $x^n - 1 = 0.$ (n even)

7. Show that the equation $x^6 - 5x^2 - x + 10 = 0$ has at least two imaginary roots. How many may it have?

8. Show that the equation $x^4 + x^2 + x - 1 = 0$ has two and only two imaginary roots.

9. Show that the equation $x^8 + 4x^3 + 2x - 10 = 0$ has six and only six imaginary roots.

10. Can you tell the nature of the roots of the equation $x^4 + ix^3 - 3ix + 4 = 0$?

302. Equations in *p*-form. If each term of the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

is divided by a_n (by hypothesis $a_n \neq 0$), we obtain the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0,$$

in which the leading coefficient is unity and $p_1 = \frac{a_{n-1}}{a_n}$, etc. An equation in this form is said to be in the ***p*-form.** For many purposes this is the most convenient form.

303. Rational Roots. *A rational root* ($\neq 0$) *of the equation* $f(x) = 0$ *when the equation is in the p-form with integral coefficients is an integer and an exact divisor of the constant term.*

Proof. Suppose that the equation $f(x) = 0$ has a root a/b where a/b ($b > 1$) is a rational fraction in its lowest terms. Then we have

$$(11) \qquad \left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0.$$

Multiplying both members of (11) by b^{n-1} we have

$$\frac{a^n}{b} + p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_nb^{n-1} = 0$$

or

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_nb^{n-1}). \tag{12}$$

The right-hand member of (12) consists of terms each of which is an integer. The left-hand member of (12) is a fraction in its lowest terms. Therefore the assumption that the fraction a/b is a root of $f(x) = 0$ leads to an absurdity.

Now suppose r ($\neq 0$) is an integral root. Then

$$r^n + p_1r^{n-1} + p_2r^{n-2} + \cdots + p_n = 0.$$

If we transpose the constant term p_n and divide by r, we obtain

$$r^{n-1} + p_1r^{n-2} + \cdots + p_{n-1} = -\frac{p_n}{r}. \tag{13}$$

Now each term of the left-hand member of (13) is an integer; hence p_n/r must be an integer, *i.e.* p_n must be exactly divisible by r.

304. To Find the Rational Roots of an Equation with Rational Coefficients. If the equation is not in the p-form with integral coefficients, reduce it to that form and then make use of the results in § 303. The following examples will explain the methods.

Example 1. Solve the equation $x^3 + 3x^2 - 4x - 12 = 0$.

By Descartes's rule of signs we know that the equation has no more than one positive root and no more than two negative roots. From the last article we know that if the equation has rational roots they are factors of 12. Thus we need only try $1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12$.

By synthetic division we have

$$\begin{array}{rrrr|r} 1 & 3 & -4 & -12 & 2 \\ & 2 & 10 & 12 & \\ \hline 1 & 5 & 6 & 0 & \end{array}$$

The depressed equation* is $x^2 + 5x + 6 = (x + 3)(x + 2) = 0$. Therefore the roots of the original equation are $2, -3, -2$.

Example 2. Solve the equation $2x^3 + x^2 + 2x + 1 = 0$.

Writing the equation in the p-form we have

$$x^3 + \tfrac{1}{2}x^2 + x + \tfrac{1}{2} = 0.$$

If we multiply the roots of this equation by k, we obtain

$$x^3 + \frac{1}{2}kx^2 + k^2x + \frac{k^3}{2} = 0.$$

If we choose k equal to 2, this equation becomes

$$(14) \qquad x^3 + x^2 + 4x + 4 = 0,$$

an equation whose roots are twice those of the original equation.

By Descartes's rule of signs equation (14) has no positive roots. Any rational roots are then negative, and are factors of 4, *i.e.* $-1, -2, -4$.

By synthetic division

$$\begin{array}{rrrr|r} 1 & 1 & 4 & 4 & -1 \\ & -1 & 0 & -4 & \\ \hline 1 & 0 & 4 & 0 & \end{array}$$

The depressed equation is $x^2 + 4 = 0$. Therefore the roots of (14) are $-1, 2i, -2i$ and the roots of the given equation are $-\frac{1}{2}, i, -i$.

EXERCISES

Solve each of the following equations.

1. $x^3 + 5x^2 + 15x + 18 = 0$.

2. $x^3 + x^2 + x + 1 = 0$.

3. $x^3 + x^2 - 4x - 4 = 0$.

4. $6x^3 + 7x^2 - 9x + 2 = 0$.

5. $6x^3 - 2x^2 + 3x - 1 = 0$.

6. $2x^4 + 3x^3 - 10x^2 - 12x + 8 = 0$.

Find the rational roots of each of the following equations.

7. $x^4 - 3x^2 - 4 = 0$.

8. $x^5 - 32 = 0$.

9. $x^4 + x^3 + x^2 + x + 1 = 0$.

10. $2x^4 - x^3 - 5x^2 + 7x - 6 = 0$.

11. $2x^4 + 2x^3 - x^2 + 1 = 0$.

12. $4x^4 - 23x^2 - 15x + 9 = 0$.

---

* If r is a root of a given equation $f(x) = 0$ and $f(x) = (x - r)Q(x)$, then the equation $Q(x) = 0$ is called the *depressed equation*.

305. The Solution of an Equation with Numerical Coefficients. The preceding articles furnish a number of methods for attacking the problem of finding the roots of an algebraic equation $f(x)=0$ with given numerical coefficients.

(1) We may examine the equation for multiple roots (§ 297).

(2) If the equation $f(x)=0$ has rational coefficients, we can find all the rational roots by a finite number of trials.

(3) When any root a has been found, we may divide $f(x)$ by $x-a$ and thus make the finding of the remaining roots depend on an equation of lower degree (the depressed equation).

306. Irrational Roots. Graphical Approximation. In order to compute approximately any one of the real irrational roots of an equation $f(x)=0$ whose coefficients are real numbers, we require first a rough approximation to the root which is to be computed. The graph of $y=f(x)$ is a powerful tool for this purpose. An example will make the method clear.

Example. Locate approximately the real roots of the equation

$$f(x)=x^5-13x^2+2x+5=0.$$

A table of corresponding values of x and $f(x)$ is as follows.

x	-2	-1	0	1	2	3
$f(x)$	-83	-11	5	-5	-11	137

Figure 252 exhibits a rough graph of this function constructed from this table. We conclude that a root of the equation lies between -1 and 0, another between 0 and 1, and a third between 2 and 3.

Moreover Descartes's rule tells us that this equation can have no more than two positive roots and no more than one negative root, since there are only two changes of sign in $f(x)$ and only one in $f(-x)$.

We have therefore located all the real roots of this equation.

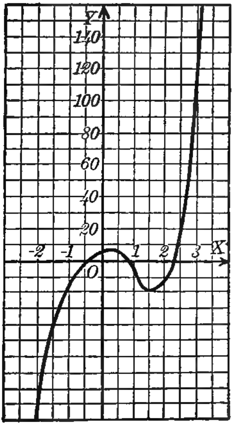

Fig. 252

A more accurate construction in the neighborhood of one of these points enables us to get a better approximation. For example, the values $x = 2.2$ and 2.3 give us respectively $y = -1.97$ and 5.21. By drawing a smooth curve through the three points corresponding to $x = 2$, 2.2, 2.3 (plotted on a large scale, Fig. 253) we may estimate the root of $f(x) = 0$ to be approximately 2.23.

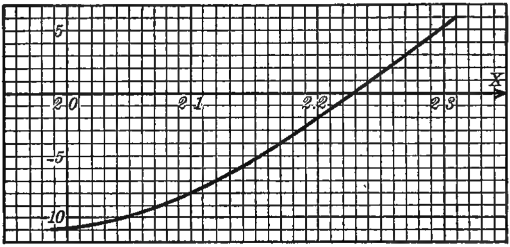

FIG. 253

307. Newton's Method of Approximation. Having found a first approximation to a root of an equation $f(x) = 0$, we may secure a better approximation by a method first suggested by Sir Isaac Newton (1642–1727). In Fig. 254 let CC' represent the graph of $y = f(x)$ in the neighborhood of a root $x = a$ of the equation. Let $OM_1 = x_1$ represent the approximation to the root found; let $M_1P_1 = y_1 = f(x_1)$. Let the tangent to the graph at $P_1(x_1, y_1)$ cut the x-axis in T. The abscissa OT will then, in general, be a much closer approximation to the desired root. The equation of the tangent at P_1 is

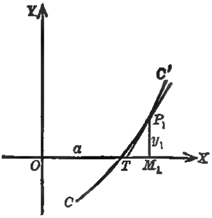

FIG. 254

$$y - f(x_1) = f'(x_1)(x - x_1). \tag{15}$$

Placing $y = 0$ and solving for x we have

$$x_2 = OT = x_1 - \frac{f(x_1)}{f'(x_1)}, \tag{16}$$

where x_2 denotes our second approximation. We have then

(17) $$x_2 = x_1 + h_1,$$

where the *correction* h_1 is given by

(18) $$h_1 = -\frac{f(x_1)}{f'(x_1)}.$$

Example. Find by Newton's method a better approximation to the root $x = 2.23$ of the equation $x^5 - 13x^2 + 2x + 5 = 0$ discussed in § 306.

$$f(x) = x^5 - 13x^2 + 2x + 5.$$
$$f'(x) = 5x^4 - 26x + 2.$$
$$f(x_1) = f(2.23) = -0.039.^*$$
$$f'(x_1) = f'(2.23) = 67.67.^*$$

Hence we have

$$h_1 = -\frac{f(x_1)}{f'(x_1)} = \frac{0.039}{67.67} = 0.00057,$$

whence $x_2 = 2.23057$.

308. The Accuracy of Newton's Method.

A question that naturally arises is: How accurate is this root, *i.e.* to how many decimal places is it correct? Taylor's theorem gives us information on this point. We have

(19) $$f(x_1 + h_1) = f(x_1) + f'(x_1)h_1 + \frac{f''(x_1)}{2!}h_1^2 + \cdots.$$

If our first approximation to the root is $x = x_1$ and h_1 is the correction,† Newton's method gives to h_1 a value which makes the sum of the first two terms of Taylor's expression vanish. Since h_1 is very small, the terms beyond the third (involving h_1^3 and higher powers of h_1) are insignificant compared with the term $\frac{f''(x_1)}{2!}h_1^2$. Hence for our purpose we may write

(20) $$f(x_1 + h_1) = \tfrac{1}{2}f''(x_1)h_1^2.$$

In the example considered above we have

$$f''(x) = 20x^3 - 26. \qquad f''(x_1) = f''(2.23) = 195.8.$$
$$h_1^2 = (0.00057)^2 = 0.00000032.$$

Hence we have

$$\tfrac{1}{2}f''(x_1)h_1^2 = f(x_1 + h_1) = 0.0000313.$$

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* Use synthetic division to get these values.

† In the example just considered $h_1 = 0.00057$.

Moreover $f'(x_1 + h_1) = f'(x_1) + f''(x_1)h_1 + \cdots,$

and in this example $f'(x_1 + h_1) = 67.67 + 0.11 = 67.78$ approximately. It follows that the new correction is about

$$h_2 = -\frac{f(x_1 + h_1)}{f'(x_1 + h_1)} < -0.000001.$$

Therefore we may conclude that $x = 2.23057$ is the root sought, to five decimal places.

EXERCISES

Find to three places of decimals the irrational roots of the following equations.

1. $x^3 + 3x + 20 = 0.$ **2.** $x^3 + 2x^2 - x + 3 = 0.$ **3.** $x^4 + x - 1 = 0.$

4. $x^3 + 4x^2 - 6 = 0.$ **5.** $x^3 + 3x^2 - 3x - 1 = 0.$

6. If x is the cosine of an angle and y is the cosine of one third of the angle, then $4y^3 = 3y + x$. Find the value of cosine of 20° to three places of decimals.

7. An open box is to be made from a rectangular piece of tin 9 × 10 inches, by cutting out equal squares from the corners and turning up the sides. How large should these squares be so that the box shall contain 59 cu. in.?

8. Find the cube root of 12; 45; −37.

309. The Relation between Roots and Coefficients. If $r_1, r_2, \cdots, r_n$ are the roots of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$, then

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = (x - r_1)(x - r_2)\cdots(x - r_n).$$

If we carry out the indicated multiplication in the right-hand member and equate the coefficients of like powers of x, we have

$$p_1 = -r_1 - r_2 - \cdots - r_n, \tag{21}$$

$$p_2 = r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_{n-1}r_n. \tag{22}$$

$$p_3 = -r_1r_2r_3 - r_1r_2r_4 - \cdots - r_{n-2}r_{n-1}r_n. \tag{23}$$

. . . .

$$p_n = (-1)^n r_1r_2\cdots r_n. \tag{24}$$

That is,

$-p_1 =$ *the sum of the roots.*

$p_2 =$ *the sum of the products of the roots taken two at a time.*

$-p_3 =$ *the sum of the products of the roots taken three at a time.*

. . . .

$(-1)^n p_n =$ *the product of all the roots.*

We have at once the following corollaries:

1. *To transform an equation into another whose roots are those of the original equation each multiplied by* m, *multiply* p_1 *by* m, p_2 *by* m^2, p_3 *by* m^3, *and so on* (§ 299).

2. *To transform an equation into another whose roots are equal to those of the original equation with their signs changed, change the signs of the alternate terms, beginning with the second.*

EXAMPLE 1. Solve the equation $2x^3 - x^2 - 8x + 4 = 0$ given that two of the roots are equal in absolute value but opposite in sign.

Let the roots be r, $-r$, and s.

Then
$$r - r + s = \tfrac{1}{2},$$
$$rs - rs - r^2 = -4,$$
$$-r^2 s = -2.$$

Therefore $s = \frac{1}{2}$ and $r = 2$ or -2, *i.e.* the roots are $\frac{1}{2}$, 2, -2.

EXERCISES

1. Solve $x^3 + x^2 - 4x - 4 = 0$, given that the sum of two of the roots is zero.

2. Solve $x^4 - 6x^3 - 9x^2 + 54x = 0$, given that the roots are in arithmetic progression.

3. Solve $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, given that the sum of two roots is 4. *Ans.* 1, 3, 5, 7.

4. Solve $4x^3 - 20x^2 - 23x - 6 = 0$, two of the roots being equal.

5. If r_1, r_2, r_3 are the roots of $x^3 - 5x^2 + 4x - 3 = 0$, find the value of each of the following expressions:

(*a*) $r_1^2 + r_2^2 + r_3^2$.

(*b*) $r_1^3 + r_2^3 + r_3^3$.

(*c*) $r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2$.

(*d*) $r_1^3 r_2 + r_1^3 r_3 + r_2^3 r_1 + r_2^3 r_3 + r_3^3 r_1 + r_3^3 r_2$.

CHAPTER XX

DETERMINANTS

310. Determinants of the Second Order. Expressions of the form $a_1b_2 - a_2b_1$, where a_1, a_2, b_1, b_2 are any numbers, arise often in mathematical analysis. Thus the area of a triangle with one vertex at the origin and the other two vertices at the points (a_1, b_1), (a_2, b_2), is equal to $\frac{1}{2}(a_1b_2 - a_2b_1)$ (§ 195). Again, the solution of a pair of simultaneous linear equations in two unknowns (§ 69) can be written as two fractions whose numerators and denominators are all of this form. (§ 311.)

The expression $a_1b_2 - a_2b_1$ may be written in the form

$$(1) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

and is then called a ***determinant of the second order.*** Such a determinant contains two rows and two columns. The numbers a_1, a_2, b_1, b_2, are called the ***elements*** of the determinant. The two elements a_1, b_2 form the so-called ***principal diagonal.***

To ***evaluate*** a determinant of the second order, *i.e.* to find what number it represents, one merely has to subtract from the product of the terms in the principal diagonal the product of the other two terms. Thus we may write

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1; \quad \begin{vmatrix} 4 & 7 \\ 3 & -6 \end{vmatrix} = (4)(-6) - (3)(7) = -45.$$

It is important to notice that each term of the expansion contains one and only one element from each row and one and only one element from each column.

311. Simultaneous Equations in Two Unknowns. Let the equations be

$$\begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases} \tag{2}$$

If we solve these equations by the usual method of elimination (§ 69), we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \tag{3}$$

provided $a_1b_2 - a_2b_1 \neq 0$. We at once recognize the fact that these results may be written in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \tag{4}$$

provided $a_1b_2 - a_2b_1 \neq 0$. The following points should be noted in the above solution.

(1) The determinants in the denominators are identical and are formed from the coefficients of x and y in the original equations.

(2) Each determinant in the numerator is formed from the determinant in the denominator by replacing by the constant terms the coefficients of the unknown whose value is sought.

Example. Solve by determinants the simultaneous equations

$$\begin{cases} 2x - y = 1, \\ 3x + 2y = 3. \end{cases}$$

Solution: $x = \dfrac{\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}} = \dfrac{5}{7}$; $y = \dfrac{\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}} = \dfrac{3}{7}$.

EXERCISES

Evaluate each of the following determinants:

1. $\begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 2a & b \\ c & -d \end{vmatrix}$.

3. $\begin{vmatrix} -\sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{vmatrix}$.

4. $\begin{vmatrix} \tan\theta & \sec\theta \\ \sec\theta & \tan\theta \end{vmatrix}$.

5. $\begin{vmatrix} \sin\theta & \cos\theta \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

6. Show that the normal form of the equation of a straight line (§ 205), may be written in the form

$$\begin{vmatrix} x & y \\ -\sin\alpha & \cos\alpha \end{vmatrix} = p.$$

Solve by the use of determinants the following pairs of equations:

7. $\begin{cases} 2x + y = 3, \\ 5x - y = 4. \end{cases}$

8. $\begin{cases} \dfrac{4x - 3y}{7} = 2, \\ \dfrac{2x}{5} = y - 7. \end{cases}$

9. $\begin{cases} x + y = 7, \\ \dfrac{x}{3} - 4y = \dfrac{1}{3}. \end{cases}$

10. $\begin{cases} x\sin\theta + y\cos\theta = \sin\theta, \\ x\cos\theta + y\sin\theta = \cos\theta. \end{cases}$

11. $\begin{cases} x + y\tan\theta = \sec^2\theta, \\ x\sec^2\theta + y\operatorname{ctn}\theta = \sec^2\theta + 1. \end{cases}$ *Ans.* 1, $\tan\theta$.

Prove the following identities and state in words what they show.*

12. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$.

13. $\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = 0.$

14. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$.

15. $\begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

16. $\begin{vmatrix} (a_1 + b_1) & b_1 \\ (a_2 + b_2) & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

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* For example, Ex. 12 shows that in a second-order determinant if the corresponding rows and columns are interchanged, the value of the determinant is not changed.

312. Determinants of the Third Order. To the square array

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{5}$$

we assign the value

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \tag{6}$$

and the name ***determinant of the third order.***

The expression (6) is known as the ***development*** or ***expansion*** of the determinant, the numbers a_1, b_1, etc., as the ***elements,*** and the elements a_1, b_2, c_3 as the ***principal diagonal.***

It is important to notice that in the development (6) each term consists of the product of three elements, one and only one from each row and one and only one from each column.

An easy way of obtaining the expansion (6) of the determinant (5) is as follows:

Form the product of each element of the first column by the second-order determinant formed by suppressing both the row and column to which the element belongs. Change the sign of the product which contains the element in the first column and the second row and take the algebraic sum of the three products.

EXAMPLE 1.
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$
$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

EXAMPLE 2.
$$\begin{vmatrix} 2 & 3 & 2 \\ -5 & 4 & 7 \\ 4 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 7 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix}$$
$$= 2(4 + 7) + 5(3 + 2) + 4(21 - 8) = 99.$$

EXAMPLE 3.
$$\begin{vmatrix} 3 & 1 & 6 \\ 0 & -1 & 4 \\ 0 & 5 & 2 \end{vmatrix} = 3\begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} = 3(-2 - 20) = -66.$$

EXERCISES

Evaluate each of the following determinants.

1. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix}$. **2.** $\begin{vmatrix} -1 & 0 & 1 \\ -2 & 2 & 0 \\ 3 & 5 & 1 \end{vmatrix}$. **3.** $\begin{vmatrix} 5 & -2 & 4 \\ 6 & -3 & 6 \\ 7 & -4 & 8 \end{vmatrix}$. **4.** $\begin{vmatrix} a & b & c \\ b & a & c \\ c & a & b \end{vmatrix}$.

5. In § 196 it was shown that the area of the triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is

$$\tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3].$$

Prove that the area of this triangle is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

6. Using the result of Ex. 5, find the area of the triangle whose vertices are

(*a*) (2, 1), (3, 1), (– 1, 7) ;
(*b*) (3, 2), (3, 6), (– 1, – 4) ;
(*c*) (0, *a*), (0, – *a*), (*b*, 0).

7. Prove that the three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ are collinear if, and only if,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

8. By means of determinants show that the three points $(a, b + c)$, $(b, c + a)$, $(c, a + b)$ are collinear.

9. By use of determinants determine whether the three points (0, 0), (1, 1), (5, 6) are collinear.

10. Prove that the equation of the straight line through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

11. By determinants find the equation of the straight line through each of the following pairs of points.

(*a*) (2, 1), (3, 7) ; (*b*) (6, 1), (2, – 1); (*c*) (7, 1), (9, 1).

12. Find by the use of determinants whether the three lines $3x - y - 7 = 0$, $2x + y + 2 = 0$, $x - y = 0$ are concurrent or not.

313. Solution of Three Simultaneous Equations. Let the equations be

$$(7)\qquad \begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

If we solve these three simultaneous equations by the usual method of elimination, we obtain,

$$(8)\qquad \begin{cases} x = \dfrac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ y = \dfrac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ z = \dfrac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \end{cases}$$

provided the denominator of each fraction is not zero. These results may be written in the form

$$(9)\qquad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Each denominator is the same determinant, which is called *the determinant of the system.* It is made up of the coefficients of x, y, z. Each determinant in the numerator is formed from the determinant in the denominator by replacing the coefficients of the unknown whose value is sought by the constant terms. Compare this rule with that given in § 311.

EXAMPLE. Solve the following equations by determinants:

$$\begin{cases} 5x - 2z = -2, \\ -3y - 4z = 7, \\ 2x - 5y = -19. \end{cases}$$

Solution.

$$x=\frac{\begin{vmatrix}-2 & 0 & -2\\ 7 & -3 & -4\\ -19 & -5 & 0\end{vmatrix}}{\begin{vmatrix}5 & 0 & -2\\ 0 & -3 & -4\\ 2 & -5 & 0\end{vmatrix}}=\frac{224}{-112}=-2;\quad y=\frac{\begin{vmatrix}5 & -2 & -2\\ 0 & 7 & -4\\ 2 & -19 & 0\end{vmatrix}}{\begin{vmatrix}5 & 0 & -2\\ 0 & -3 & -4\\ 2 & -5 & 0\end{vmatrix}}=\frac{-336}{-112}=3;$$

$$z=\frac{\begin{vmatrix}5 & 0 & -2\\ 0 & -3 & 7\\ 2 & -5 & -19\end{vmatrix}}{\begin{vmatrix}5 & 0 & -2\\ 0 & -3 & -4\\ 2 & -5 & 0\end{vmatrix}}=\frac{448}{-112}=-4.$$

EXERCISES

Expand each of the following determinants:

1. $\begin{vmatrix}2 & 3 & 3\\ 4 & -1 & 2\\ -1 & 4 & 1\end{vmatrix}$. **2.** $\begin{vmatrix}-7 & 1 & 2\\ 2 & -3 & -6\\ 4 & 2 & 4\end{vmatrix}$. **3.** $\begin{vmatrix}x & z & z\\ x & y & y\\ y & z & x\end{vmatrix}$.

4. $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{vmatrix}$. **5.** $\begin{vmatrix}a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}$.

Solve by determinants each of the following sets of equations:

6. $\begin{cases}4x+5y+2z=20,\\ 3x-3y+5z=12,\\ 5x+2y-4z=-3.\end{cases}$ **7.** $\begin{cases}3x+y-z=3,\\ x+y+z=7,\\ 2x+4y+z=12.\end{cases}$

Ans. (1, 2, 3).

8. $\begin{cases}x+y+z=1,\\ ax+by+cz=d,\\ a^2x+b^2y+c^2z=d^2.\end{cases}$ **9.** $\begin{cases}ax+y-z=a^2+a-1,\\ -x+ay+z=a^2-a+1,\\ x-y+az=a.\end{cases}$

Ans. $(a, a, 1)$.

10. Solve the equation

$$\begin{vmatrix}x & 2 & 1\\ 3 & x & -1\\ 2 & 4 & 5\end{vmatrix}=0.$$

11. Solve for x and y the simultaneous equations

$$\begin{vmatrix}x+1 & 2 & 1\\ 3 & x & -1\\ y & 2 & 1\end{vmatrix}=0,\quad \begin{vmatrix}x & 1 & 3\\ -2 & 1 & -2\\ y & 1 & 4\end{vmatrix}=0.$$

12. Evaluate the determinant

$$\begin{vmatrix} \sin\alpha & \cos\beta & 1 \\ \cos\alpha & \sin\beta & 1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Prove the following identities and express in words what they prove. See Ex. 12–16, pp. 477.

13. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_3 & b_2 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

14. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

15. $$\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0.$$

16. $$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

17. $$\begin{vmatrix} (a_1 + b_1) & b_1 & c_1 \\ (a_2 + b_2) & b_2 & c_2 \\ (a_3 + b_3) & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

314. Inversions. Let us consider the permutations of a set of objects, such as letters or numbers, and let us fix a certain particular order of the objects which we shall designate as the *normal order*. An inversion is said to occur in any permutation when an object is followed by one which in the normal order precedes it. Thus if *abcd* is the normal order, then there are two inversions in *badc*. If 1234 is the normal order, then there are three inversions in 1432.

THEOREM. *If in a given permutation, two objects are interchanged, the number of inversions with respect to the normal order is increased or decreased by an odd number.*

Let us consider the permutations $XrsY$ and $XsrY$, where X and Y denote the groups of objects which precede and follow the interchanged objects r and s. Any inversion in X and Y and any inversion due to the fact that X, r, s precede Y are common to $XrsY$ and $XsrY$. Therefore, the number of inversions in $XrsY$ is equal to the number in $XsrY$ increased or decreased by 1 (according as rs is or is not in the normal order).

Now let us consider two objects such as r and s separated by i objects. If the objects r and s are interchanged, the number of inversions is still changed by an odd number. For, by $i + 1$ interchanges of adjacent pairs the object r can be brought into the position immediately following s, and by i further interchanges of adjacent pairs, s may be brought to occupy the position formerly held by r. Each of these $(i+1)+i=2i+1$ interchanges of adjacent pairs has increased or decreased the number of inversions by 1. Hence the net result of these $2i+1$ interchanges has increased or decreased the number of inversions by an odd number.

315. Determinants of the *n*th Order.

The square array

$$(10)\qquad \begin{vmatrix} a_1 & b_1 & . & . & . & q_1 \\ a_2 & b_2 & . & . & . & q_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_n & b_n & . & . & . & q_n \end{vmatrix}$$

of n^2 elements, such as we have considered for the cases $n=2$, $n=3$, is called a ***determinant of the nth order*** and will be denoted by the Greek letter Δ. *This determinant will be understood to stand for the algebraic sum of all the different products of n factors each that can be formed by taking one and only one element from each row and one and only one element from each column, and giving to each such product a positive or negative sign according as the number of inversions of the subscripts (normal order* $1, 2, \cdots, n$*) is even or odd, when the letters have the normal order* $ab \cdots q$.

It should be noted that from the remarks in § 314 it follows that if we arrange the elements in any product so that the subscripts are in normal order, we can determine the sign of each term, by making it positive or negative according as the number of inversions of the letters is even or odd.

316. Properties of Determinants.

THEOREM 1. *The expansion of a determinant of order n contains n ! terms.*

PROOF. There are as many terms in the expansion of a determinant of the nth order as there are permutations of the subscripts $1, 2, 3, \cdots, n$. But this number is $n!$ (§ 269).

THEOREM 2. *If each element of any row or column is multiplied by any constant m, the value of the determinant is multiplied by m.*

PROOF. Since by the definition of a determinant, each term of the expansion must contain one and only one element from each row and each column, the factor m will appear once and only once in each term of the expansion. If m is factored out of this expansion, the remaining factor is the expansion of the original determinant.

ILLUSTRATION.

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = ma_1b_2c_3 + ma_2b_3c_1 + ma_3b_1c_2 - ma_1b_3c_2 - ma_2b_1c_3 - ma_3b_2c_1$$

$$= m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

THEOREM 3. *The value of a determinant is not changed if rows and columns are interchanged, so that the first row becomes the first column, the second row the second column, and so on.*

This follows at once from the definition of the determinant and the paragraph immediately following it (§ 315).

THEOREM 4. *If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed.*

ILLUSTRATION. See Ex. 14, p. 477, and Ex. 14, p. 482.

PROOF: Since by Theorem 3 rows and columns may be interchanged without affecting the value of the determinant, we need only consider the interchange of two rows. First, if two adjacent rows are interchanged, the order of the letters in the principal diagonal and in each term of the development is left unchanged. However two adjacent subscripts in each term of the expansion are interchanged, and hence the sign of every term is changed. Why?

Next consider the effect of interchanging two rows separated by k intermediate rows. By k interchanges of adjacent rows, the lower row can be brought just below the upper one. Now the upper row can be brought into the original position of the lower row by $k + 1$ further interchanges of adjacent rows. Therefore interchanging the two rows is equivalent to $2k + 1$ interchanges of adjacent rows. But $2k + 1$ is an odd number and therefore this process changes the sign of the determinant.

THEOREM 5. *If two rows or two columns of a determinant are identical, the value of the determinant is zero.*

PROOF: Let Δ be the value of the determinant and let the two identical rows or columns be interchanged. Then, by Theorem 4, the value of the resulting determinant is $-\Delta$. But since the rows or columns which were interchanged were identical, the value of the determinant is left unchanged. That is to say, $\Delta = -\Delta$ or $2\Delta = 0$, or $\Delta = 0$.

COROLLARY. *If all the elements in any row or column are the same multiples of the corresponding elements in any other row or column, then the value of the determinant is zero.*

317. Minors. If we suppress the row and the column in which any given element appears, the determinant formed by the remaining elements is called the minor of that element.

Illustration. In the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the minor of a_2 is

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix};$$

and the minor of c_3 is

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

The minor of a_1 is denoted by A_1, of b_j by B_j, etc.

EXERCISES

1. Prove that $\begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 5 \\ 4 & 4 & 9 \end{vmatrix} = 0.$ **2.** Prove that $\begin{vmatrix} 3 & 5 & 8 \\ 1 & 2 & 5 \\ 2 & 4 & 10 \end{vmatrix} = 0.$

3. Prove that $\begin{vmatrix} 4 & 5 & 6 \\ 2 & 1 & 5 \\ 1 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 1 \\ 5 & 1 & 5 \\ 6 & 5 & 3 \end{vmatrix}.$

4. Prove that $\begin{vmatrix} 3 & 4 & 5 \\ 2 & 4 & 1 \\ 8 & 4 & 5 \end{vmatrix} = -\begin{vmatrix} 4 & 3 & 5 \\ 4 & 2 & 1 \\ 4 & 8 & 5 \end{vmatrix}.$

5. Prove that $\begin{vmatrix} 4 & 1 & 6 & 5 & 3 \\ 3 & -1 & 5 & -1 & -3 \\ 2 & 2 & 2 & -3 & 6 \\ 1 & 3 & 3 & 2 & 9 \\ 5 & 4 & -1 & 1 & 12 \end{vmatrix} = 0.$

6. Prove that $\begin{vmatrix} 26 & 9 & -5 \\ 28 & 18 & 10 \\ 30 & 3 & -25 \end{vmatrix} = 30 \begin{vmatrix} 13 & 3 & -1 \\ 14 & 6 & 2 \\ 15 & 1 & -5 \end{vmatrix}.$

7. How many inversions are there in the arrangement 4213765 if the normal order is 1234567?

8. How many inversions are there in the arrangement 45321 if the normal order is 42315?

9. Find the value of the minor of 5, of 6, of 7, for the determinant

$$\begin{vmatrix} 4 & 5 & 1 \\ 3 & 6 & 2 \\ 2 & 7 & 8 \end{vmatrix}.$$

10. Write down the minor of a_3, of c_2, of b_4, for the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

11. Show that

$$\begin{vmatrix} 1 & 2 & 5 & -1 \\ 3 & 3 & 6 & 2 \\ 4 & 2 & 7 & 3 \\ 5 & 1 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 4 & 5 \\ 2 & 3 & 2 & 1 \\ 5 & 6 & 7 & 5 \\ -1 & 2 & 3 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 \\ 5 & 5 & 7 & 6 \\ -1 & 4 & 3 & 2 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & 2 & 3 \\ 1 & 5 & 4 & 3 \\ -1 & 4 & 3 & 2 \\ 5 & 5 & 7 & 6 \end{vmatrix}.$$

318. Additional Theorems.

— The following theorems will be found useful in evaluating determinants.

THEOREM 6. LAPLACE'S EXPANSION. *If the product of each element in any row or column by its corresponding minor be given a positive or negative sign according as the sum of the number of the row and the number of the column containing the element is even or odd, then the algebraic sum of these products is the value of the determinant.*

PROOF: First, it is evident that in the development of the determinant, A_1 is the coefficient of a_1. For A_1 is a determinant of order $n-1$ in the elements $a_2, \cdots, a_n$, and its expansion contains a term for each permutation of $2, 3, \cdots, n$. Moreover, the signs of the terms are correct; for, the number of inversions is not changed by prefixing a_1.

Second, let us consider the element e situated in the ith row and the jth column. We can bring this element to the leading position, *i.e.* first row and first column, by $i-1$ transpositions of rows and $j-1$ transpositions of columns, *i.e.* by $i+j-2$ transpositions in all. Therefore the sign of the determinant will have been changed $i+j-2$ times. That is, if $i+j$ is an even number, the sign of the determinant is left unchanged; while if $i+j$ is an odd number, the sign of the determinant is changed. Now that the element under consideration is in the leading position, we know from the first step that its coefficient is its minor. Since the relative positions of the elements not in the ith row or the jth column are not effected by these transpositions, the minor of the element in its original position is the same as the minor of the element when it is in the leading position.

Hence the coefficient of the element e, which is situated in the ith row and the jth column, is $(-1)^{i+j}E$, where E is the minor of the element e.

Corollary. *If in the development of a determinant by minors with respect to a certain column (row) the elements of this column (row) are replaced by the corresponding elements of some other column (or row), the resulting expression vanishes.*

Illustration.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4.$$

We wish to show that, for example, $b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4$ is zero. This expression is zero, for we have replaced the column of a's by the column of b's and hence the determinant has two columns identical. The same proof applies to a determinant of order n.

Theorem 7. *If each of the elements of any row or column of a determinant consists of the sum of two numbers, the determinant may be expressed as the sum of two determinants.*

Proof: Let

$$\begin{vmatrix} (a_1 + a'_1) & b_1 \cdots q_1 \\ (a_2 + a'_2) & b_2 \cdots q_2 \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ (a_n + a'_n) & b_n \cdot\cdot\ q_n \end{vmatrix}$$

be the given determinant. Expanding in terms of the first column we have

$$(a_1 + a'_1)A_1 - (a_2 + a'_2)A_2 + (a_3 + a'_3)A_3 + \cdots + (-1)^{n-1}(a_n + a'_n)A_n$$

or

$$[a_1A_1 - a_2A_2 + a_3A_3 + \cdots + (-1)^{n-1}a_nA_n]$$
$$+ [a'_1A_1 - a'_2A_2 + a'_3A_3 + \cdots + (-1)^{n-1}a'_nA_n],$$

i.e.

$$\begin{vmatrix} a_1 & b_1 \cdots q_1 \\ a_2 & b_2 \cdots q_2 \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_n & b_n \cdots q_n \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 \cdots q_1 \\ a'_2 & b_2 \cdots q_2 \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a'_n & b_n \cdots q_n \end{vmatrix}.$$

Theorem 8. *If to the elements in any row (or column) be added the corresponding elements of any other row (or column) each multiplied by a given number m, the value of the determinant is unchanged.*

The proof of this theorem follows easily from Theorems 7, 5, and 2.

319. The Evaluation of Determinants. We are now in a position to expand a determinant of any order. The following examples will illustrate the methods employed.

EXAMPLE 1. Expand

$$\Delta = \begin{vmatrix} 25 & 26 & 27 \\ 26 & 27 & 28 \\ 27 & 28 & 29 \end{vmatrix}.$$

Multiply the first column by -1 and add it to the second and third columns. It gives

$$\Delta = \begin{vmatrix} 25 & 1 & 2 \\ 26 & 1 & 2 \\ 27 & 1 & 2 \end{vmatrix}.$$

By the corollary of Theorem 5, the value of this determinant is 0.

EXAMPLE 2. Expand the determinant

$$\Delta = \begin{vmatrix} 2 & -1 & 5 & 1 \\ 1 & 4 & 6 & 3 \\ 4 & 2 & 7 & 4 \\ 3 & 1 & 2 & 5 \end{vmatrix}.$$

We seek to transform this determinant in such a way as to make all the elements but one in some row or column 0. The second column looks most promising. We accordingly add 4 times the first row to the second row (this replaces the 4 in the second row by 0); we then add 2 times the first row to the third row (Why?); and then add the first row to the fourth row (Why?) These operations give

$$\Delta = \begin{vmatrix} 2 & -1 & 5 & 1 \\ 9 & 0 & 26 & 7 \\ 8 & 0 & 17 & 6 \\ 5 & 0 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 9 & 26 & 7 \\ 8 & 17 & 6 \\ 5 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 26 & 7 \\ 2 & 17 & 6 \\ -1 & 7 & 6 \end{vmatrix}^{*}.$$

The last determinant may be still further simplified as follows:

$$\Delta = \begin{vmatrix} 2 & 26 & 7 \\ 2 & 17 & 6 \\ -1 & 7 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 40 & 19 \\ 0 & 31 & 18 \\ -1 & 7 & 6 \end{vmatrix}$$

$$= -\begin{vmatrix} 40 & 19 \\ 31 & 18 \end{vmatrix} = -\begin{vmatrix} 9 & 1 \\ 31 & 18 \end{vmatrix}$$

$$= -(162-31) = -131.$$

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* This determinant is obtained from the preceding by subtracting the elements of the last column from those of the first.

EXERCISES

Evaluate the following determinants.

1. $\begin{vmatrix} 14 & 13 & -12 \\ 17 & 16 & 17 \\ 25 & 24 & -18 \end{vmatrix}$.

2. $\begin{vmatrix} 34 & 23 & 12 \\ 23 & 34 & 21 \\ 14 & 35 & 26 \end{vmatrix}$.

3. $\begin{vmatrix} 18 & 26 & 24 \\ 29 & 39 & 49 \\ 37 & 35 & 11 \end{vmatrix}$.

4. $\begin{vmatrix} 2 & -2 & 1 & 1 \\ 1 & -1 & 4 & 2 \\ 2 & -2 & 1 & -1 \\ 0 & 2 & 1 & -1 \end{vmatrix}$.

5. $\begin{vmatrix} 3 & 4 & -2 & 5 \\ 4 & -3 & 8 & -4 \\ 2 & 8 & 3 & 0 \\ 1 & 0 & 4 & 1 \end{vmatrix}$.

6. $\begin{vmatrix} 23 & 24 & 25 & 26 \\ 12 & 13 & 14 & 15 \\ 32 & 33 & 34 & 35 \\ 2 & 2 & 2 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} a & b & c+d \\ a & c & b+d \\ a & d & b+c \end{vmatrix}$.

8. $\begin{vmatrix} 1 & 2a & a^2 \\ 1 & a+b & ab \\ 1 & 2b & b^2 \end{vmatrix}$.

9. $\begin{vmatrix} a & b & c & d \\ b & x & 0 & 0 \\ c & 0 & y & 0 \\ d & 0 & 0 & z \end{vmatrix}$.

10. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

11. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$.

12. $\begin{vmatrix} a & b & c & d \\ -a & b & c & d \\ -a & -b & c & d \\ -a & -b & -c & d \end{vmatrix}$.

13. Prove that if a determinant whose elements are rational integral functions of some variable, as y, vanishes when $y = b$, then $y - b$ is a factor of the determinant.

[HINT: Use the corollary of theorem 6.]

14. Solve by factoring Examples 10, 11, 12.

15. Factor into two factors

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

16. Factor

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}.$$

320. Solution of a System of Linear Equations. Suppose we have n linear equations in n unknowns and we desire their solution. Let the equations be

$$(11)\qquad \begin{cases} a_1x_1 + b_1x_2 + c_1x_3 + \cdots + p_1x_n = q_1 \\ a_2x_1 + b_2x_2 + c_2x_3 + \cdots + p_2x_n = q_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_nx_1 + b_nx_2 + c_nx_3 + \cdots + p_nx_n = q_n \end{cases}$$

Let Δ be the determinant of the cofficients of the unknowns, *i.e.*

$$(12)\qquad \Delta = \begin{vmatrix} a_1 & b_1 & \cdots & p_1 \\ a_2 & b_2 & \cdots & p_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & \cdots & p_n \end{vmatrix}$$

The determinant Δ is called *the determinant of the system.* Multiply the equations by A_1, $-A_2$, A_3, $-A_4$, etc., respectively, and add the results. Then we have

$$(13)\quad x_1(a_1A_1 - a_2A_2 \cdots) + x_2(b_1A_1 - b_2A_2 \cdots) + \cdots + x_n(p_1A_1 - p_2A_2 \cdots) = q_1A_1 - q_2A_2 \cdots.$$

From the corollary of Theorem 6 it follows that the coefficient of x_1 is Δ and that the coefficients of the other unknowns are zero. Moreover, the right-hand member of (13) is the expansion of Δ if we replace the column of a's by the column of constant terms. This determinant will be denoted by Δ_{aq}. Therefore we may write

$$\Delta \cdot x_1 = \Delta_{aq},$$

or

$$x_1 = \frac{\Delta_{aq}}{\Delta},$$

provided $\Delta \neq 0$.

Similarly

$$x_2 = \frac{\Delta_{bq}}{\Delta},\ \cdots,\ x_n = \frac{\Delta_{pq}}{\Delta},$$

provided $\Delta \neq 0$.

It will be noticed that this is a direct extension of the methods employed in §§ 311, 313. The result may be stated in words as follows. The value of any unknown is equal to a fraction whose denominator is the determinant of the system and whose numerator is the determinant obtained from the former by replacing the coefficients of the unknown sought by the column of constant terms.

321. The Case $\Delta = 0$. The previous methods show that, even if $\Delta = 0$, we can derive from the given equations the relations

$$\Delta \cdot x_1 = \Delta_{aq},\ \Delta \cdot x_2 = \Delta_{bq},\ \cdots,\ \Delta \cdot x_n = \Delta_{pq}.$$

Now if $\Delta = 0$, these relations would imply that

$$\Delta_{aq} = 0,\ \Delta_{bq} = 0,\ \cdots,\ \Delta_{pq} = 0.$$

But it is easy to write down a system in which $\Delta = 0$ and one or more of the Δ_{aq}, $\Delta_{bq} \cdots$ are not zero. Such a system is then clearly inconsistent and has no solution. For example, $2x_1 + x_2 = 1$, $2x_1 + x_2 = 2$.

If $\Delta_{aq} = \Delta_{bq} = \cdots = \Delta_{pq} = 0$, the system may be consistent but the unknowns $x_1, x_2, \cdots, x_n$ are not then completely determined. For example, $2x_1 + x_2 = 1$, $4x_1 + 2x_2 = 2$.

A complete discussion of this case is beyond the scope of this book.*

322. Consistent Equations. Equations which have a common solution are called *consistent.* Consider the three equations in two unknowns x and y:

$$a_1x + b_1y + c_1 = 0. \tag{14}$$

$$a_2x + b_2y + c_2 = 0. \tag{15}$$

$$a_3x + b_3y + c_3 = 0. \tag{16}$$

Two cases arise according as to whether a pair of the three equations has a single or an infinite number of solutions.

Case 1. *A single solution.* In order that these three equations be consistent it is necessary that

$$x = -\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0\right] \tag{17}$$

satisfy equation (16), *i.e.* that

$$-a_3\begin{vmatrix} c_1 & b_1 \\ c^2 & b^2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

or its equivalent

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

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* Those interested in this problem will find a complete discussion in Bôcher, *Higher Algebra*, Chapter IV.

Case 2. *An infinite number of solutions.* In this case

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

and hence, by the corollary of theorem 5, the above determinant must equal zero. Therefore, *in order that three linear equations in two unknowns have a common solution, it is necessary that the determinant of the coefficients of the unknowns and the known terms vanish.*

Extending this result to n linear equations in $n-1$ unknowns, we have *a necessary condition that n linear equations in $n-1$ unknowns be consistent is that the determinant formed from the coefficients of the unknowns and the known terms must vanish.*

It must be clearly understood that the vanishing of the above determinant is only a *necessary* and not a sufficient condition that the equations be consistent. For example, the system

$$\begin{cases} 2x+\ \ y-1=0, \\ 2x+\ \ y+5=0, \\ 4x+2y+3=0, \end{cases}$$

gives

$$\Delta=\begin{vmatrix} 2 & 1 & -1 \\ 2 & 1 & 5 \\ 4 & 2 & 3 \end{vmatrix}=0,$$

but the equations are inconsistent, for any pair are inconsistent.

EXERCISES

Solve the following systems of equations by means of determinants:

1. $\begin{cases} 2x-y-z=0, \\ 3x+y+z=5, \\ 2x-3y-v=-2, \\ 2x+3v=5. \end{cases}$

2. $\begin{cases} -x+y+z=2m, \\ x-y+z=2n, \\ x+y-z=2p. \end{cases}$

3. $\begin{cases} 3y-4x-2z+w=-21, \\ x+7y+z-w=13, \\ y-2x-3z+2w=14, \\ 3x+5y-5z+3w=11. \end{cases}$

4. $\begin{cases} x+y+w=6, \\ x+y+z=7, \\ y+z+w=8, \\ x+z+w=9. \end{cases}$

5. $\begin{cases} x+2y-z+3w=-10, \\ x+3y-2z-4w=1, \\ 2x-y-3z+5w=3, \\ 3x-y-z-2w=18. \end{cases}$

Determine whether the following systems of equations are consistent:

6. $\begin{cases} 2x+y+1=4, \\ x+y-2=0, \\ 3x-6y-2=0. \end{cases}$ 7. $\begin{cases} 2x+y-1=0, \\ 3x-2y+7=0. \\ 4x+y-2=0. \end{cases}$ 8. $\begin{cases} x-y-2=0, \\ x-y+7=0, \\ 3x+y-2=0. \end{cases}$

9. Find k so that the following equations are consistent:

$$\begin{aligned} 2x+y-3&=0, \\ 3x-y&=2, \\ x+y+k&=0. \end{aligned}$$

MISCELLANEOUS EXERCISES

1. Prove that the equation of the circle that passes through the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\begin{vmatrix} (x^2+y^2) & x & y & 1 \\ (x_1^2+y_1^2) & x_1 & y_1 & 1 \\ (x_2^2+y_2^2) & x_2 & y_2 & 1 \\ (x_3^2+y_3^2) & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

2. Prove that $ax^2+by^2+2hxy+2fx+2gy+c$ is the product of two linear functions if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

3. Prove that a necessary condition that the three lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$, be concurrent is that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Is this condition also sufficient?

4. Prove that the locus of the equation $ax+by+c=0$ is a straight line.

[HINT: Let (x_1, y_1), (x_2, y_2) be any two fixed points on the locus and (x, y) any other point on the locus. Then we have $ax_1+by_1+c=0$, $ax_2+by_2+c=0$, $ax+by+c=0$. Since these equations are consistent, the determinant of the coefficient is zero.]

PART V. FUNCTIONS OF TWO VARIABLES

SOLID ANALYTIC GEOMETRY

CHAPTER XXI

LINEAR FUNCTIONS

THE PLANE AND STRAIGHT LINE

323. Introduction. Thus far the only functions which we have represented geometrically are those of the form $y = f(x)$, *i.e.* functions of a single independent variable x. Such functions, in general, were seen to represent a curve in the (x, y) plane. We shall now study functions of the form $z = f(x, y)$, *i.e.* functions of two independent variables x and y. In order to carry out this investigation it is necessary to set up a coördinate system in three dimensions.

324. Orthogonal Projections. The ***orthogonal projection of a point*** P upon a plane α (Fig. 255) is the foot P' of the perpendicular drawn from P to α. The ***orthogonal projection of a segment*** PQ upon α is the segment PQ' joining the projections of P and Q upon α.

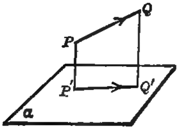

FIG. 255

The ***orthogonal projection of a point*** P upon a line l is the foot P' of the perpendicular drawn from P to l. The ***orthogonal projection of a segment*** PQ upon the line l is the segment $P'Q'$ joining the projections of P and Q upon l.

325. Rectangular Coördinates in Space. Consider three mutually perpendicular planes intersecting in the lines $X'X$, $Y'Y$, $Z'Z$. These lines are themselves mutually perpendicular. The three planes are known as the *coördinate planes* and their three lines of intersection as the *coördinate axes.* The planes are known as the ***xy-plane***, ***yz-plane***, ***xz-plane***, and the axes as the ***x-axis***, ***y-axis***, ***z-axis***. The point O which is common to the three planes and also to the three axes, is called the ***origin.*** The positive directions of these axes are usually taken as indicated by the arrows in Fig. 256.

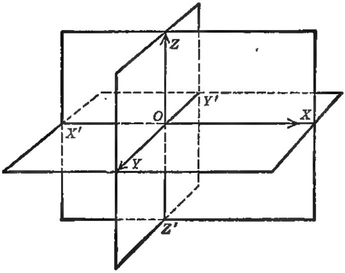

FIG. 256

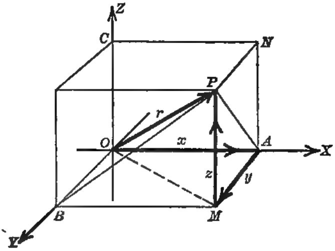

FIG. 257

Let P be any point in space, and let us consider the segment OP. The numbers representing the projections of OP on the three axes we call the coördinates of P and denote them by x, y, and z. In Fig. 257, $x = OA$, $y = OB$, $z = OC$.

Conversely, any three real numbers x, y, z may be considered as the coördinates of a point P. Why? If M is the foot of the perpendicular dropped from P on the xy-plane, and A is the foot of the perpendicular dropped from M on the x-axis, the coördinates of P are $x = OA$, $y = AM$, $z = MP$.

The eight portions of space separated by the coördinate planes are called ***octants***. From the preceding definitions it follows that the signs of the coördinates of a point P in any octant are as follows:

(*a*) x is positive or negative according as P lies to the right or left of the yz-plane;

(*b*) y is positive or negative according as P lies in front or back of the xz-plane;

(*c*) z is positive or negative according as P lies above or below the xy-plane.

EXERCISES

1. What are the coördinates of the origin?

2. What is the z coördinate of any point in the xy-plane?

3. What are the x and y coördinates of any point on the z-axis?

4. What is the locus of points for which $x = 0$? for which $y = 0$? for which $z = 0$?

5. What is the locus of points for which $x = 0$ and $y = 0$?

6. What is the locus of points for which $y = 0$ and $z = 0$?

7. What is the locus of points for which $z = 0$ and $x = 0$?

8. What is the locus of points for which $x = 2$ and $y = 2$?

9. If $P(x, y, z)$ is any point in space, find

(*a*) its distance from the xy-plane;
(*b*) its distance from the yz-plane;
(*c*) its distance from the xz-plane;
(*d*) its distance from the x-axis;
(*e*) its distance from the y-axis;
(*f*) its distance from the z-axis.

10. Describe the positions of each of the following points: $(2, -8, 3)$; $(-2, 3, -5)$; $(3, 3, -3)$; $(-4, -7, -9)$.

11. Plot the following points: $(2, 1, 3)$; $(4, -1, -2)$; $(0, 0, -3)$; $(3, 1, 1)$; $(-1, -1, -1)$; $(1, 0, 1)$; $(-1, 2, -1)$; $(1, -1, 0)$; $(4, -1, -1)$.

12. Find the distance from the origin to the point $P(x, y, z)$.

13. A point P moves so that its distance from the origin is always equal to 4. Find the equation of the locus of P.

14. Show that the points (x, y, z) and $(-x, y, z)$ are symmetric with respect to the yz-plane.

15. A rectangular parallelepiped has three of its faces in the coördinate planes. Find the coördinates of its vertices, assuming that the dimensions of the parallelepiped are a, b, c.

326. Directed Segments. We shall define the *angle between two directed lines l and m* which do not meet, to be the angle between two similarly directed lines l' and m' which do meet (Fig. 258).

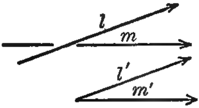

FIG. 258

THEOREM I. *If AB is a directed segment on a line l, which makes an angle θ with the directed line l', then*

(1) $$\mathbf{Proj}_{l'}\,AB = AB\cos\theta.$$

PROOF: Through A' (Fig. 259) draw l_1 parallel to l and let B_1 be the projection of B on l_1. Then, by definition, the angle between l and l' is the same as the angle between l_1 and l'. It follows from § 135 that

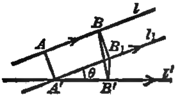

FIG. 259

$$A'B' = A'B_1\cos\theta,$$

or $$A'B' = AB\cos\theta,$$

since $$A'B_1 = AB.$$

THEOREM II. *The projection on a directed line s of a broken line made up of the segments A_1A_2, A_2A_3, A_3A_4, $\cdots$, $A_{n-1}A_n$, is equal to the projection on s of the segment A_1A_n.*

The proof of this theorem is left as an exercise. See § 136.

COROLLARY. *If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then*

(2) $$\begin{cases} x_2 - x_1 = \mathbf{Proj}_x\,P_1P_2, \\ y_2 - y_1 = \mathbf{Proj}_y\,P_1P_2, \\ z_2 - z_1 = \mathbf{Proj}_z\,P_1P_2. \end{cases}$$

EXERCISES

1. Find the projections upon the coördinate axes of the sides of the polygon $ABCDEF$ whose vertices are $A(0, 0, 0)$, $B(1, -6, 4)$, $C(-2, 4, -1)$, $D(3, -1, 2)$, $E(2, 1, 4)$, $F(1, 1, 1)$.

2. The projections of the segment MP upon the coördinate axes are 4, 3, -1 respectively. If M is $(2, -1, 3)$, find the coördinates of P.

327. Direction Cosines of a Line. Let l be any directed line and l' a line through the origin having the same direction. If l' makes angles α, β, and γ with the x, y, and z axes respectively, then, by definition, l makes the same angles with these axes. These angles are known as the ***direction angles*** of the line l, while their cosines are called the ***direction cosines*** of l.

Reversing the direction of a line changes the signs of the direction cosines of the line. For reversing the direction of a line changes α, β, γ into $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$, respectively; and by § 122 $\cos(\pi - \theta) = -\cos\theta$.

THEOREM. *The sum of the squares of the direction cosines of a line is equal to unity.*

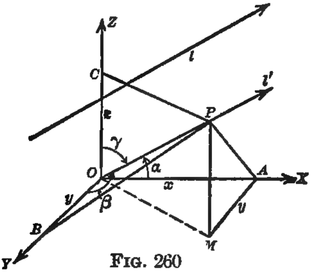

FIG. 260

PROOF. Let $P(x, y, z)$ be any point on l' (Fig. 260). Then, we have

$$\left\{\begin{aligned} x &= OP\cos\alpha, \\ y &= OP\cos\beta, \\ z &= OP\cos\gamma. \end{aligned}\right. \tag{3}$$

Therefore,

$$x^2 + y^2 + z^2 = \overline{OP}^2[\cos^2\alpha + \cos^2\beta + \cos^2\gamma].$$

Since $x^2 + y^2 + z^2 = \overline{OP}^2$,* it follows that

$$\boldsymbol{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.} \tag{4}$$

---

* $\overline{OM}^2 = x^2 + y^2$ and $\overline{OP}^2 = z^2 + \overline{OM}^2 = z^2 + x^2 + y^2$.

Any three numbers l, m, n, (not all zero) are proportional to the direction cosines of some line; for, $P(l, m, n)$ is a point and the direction cosines of OP are

$$\cos\alpha = \frac{l}{\pm\sqrt{l^2+m^2+n^2}},$$

(5) $$\cos\beta = \frac{m}{\pm\sqrt{l^2+m^2+n^2}},$$

$$\cos\gamma = \frac{n}{\pm\sqrt{l^2+m^2+n^2}}.$$

The direction cosines of OP are evidently proportional to l, m, n, and they may be found by dividing l, m, and n, respectively, by $\pm\sqrt{l^2+m^2+n^2}$.

328. The Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Let the direction angles of the segment P_1P_2 be α, β, γ. Projecting P_1P_2 upon the axes, we have, from the corollary of § 326,

$$P_1P_2\cos\alpha = x_2 - x_1,\quad P_1P_2\cos\beta = y_2 - y_1,\quad P_1P_2\cos\gamma = z_2 - z_1.$$

Squaring and adding we have, by the theorem of § 327,

$$\overline{P_1P_2}^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

Therefore,

(6) $$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

EXERCISES

1. Find the length and the direction cosines of the segment P_1P_2, when

(*a*) P_1 is $(2, 3, 4)$ and P_2 is $(-1, 0, 5)$;
(*b*) P_1 is $(-1, 2, -7)$, and P_2 is $(4, 1, 4)$;
(*c*) P_1 is $(4, 7, 1)$, and P_2 is $(1, -2, -7)$.

2. Prove that the triangle whose vertices are $A(m, n, p)$, $B(n, p, m)$, $C(p, m, n)$ is equilateral.

3. Find the direction cosines of a line which are proportional to 4, 7, 1.

4. Find the length of a line-segment whose projections on the coordinate axes are 4, 7, 2.

329. The Angle between Two Directed Lines. If $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are the direction angles of two directed lines l_1 and l_2 the angle θ between them may be determined as follows.

Draw the lines l'_1 and l'_2 through the origin, parallel to the given lines (Fig. 261). Then the angle between l'_1 and l'_2 is θ.

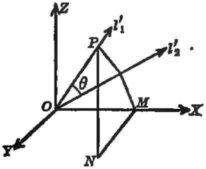

FIG. 261

If $P(x, y, z)$ is any point on l'_1, then, by Theorem II of § 326, we have

$$\text{Proj}_{l'_2}\, OP = \text{Proj}_{l'_2}\, OMNP,$$

i.e.
$$OP\cos\theta = OM\cos\alpha_2 + MN\cos\beta_2 + NP\cos\gamma_2.$$

But,

$$OM = OP\cos\alpha_1, \quad MN = OP\cos\beta_1, \quad NP = OP\cos\gamma_1.$$

Therefore,

$$(7) \qquad \cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2.$$

We shall assume that θ is the smallest positive angle satisfying equation (7).

330. Parallel and Perpendicular Lines. If two lines are parallel and extend in the same direction, they are parallel to and agree in direction with the same line through the origin.

Therefore, if $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are the direction angles of the two lines, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$; and we may write

$$(8) \qquad \cos\alpha_1 = \cos\alpha_2, \quad \cos\beta_1 = \cos\beta_2, \quad \cos\gamma_1 = \cos\gamma_2.$$

Conversely, if relations (8) are satisfied, the given lines are parallel and extend in the same direction. Why?

If the two lines are *parallel* but extend in opposite directions, we have $\alpha_1 = \pi - \alpha_2$, $\beta_1 = \pi - \beta_2$, $\gamma_1 = \pi - \gamma_2$, and therefore,

$$(9)\quad \cos \alpha_1 = -\cos \alpha_2, \quad \cos \beta_1 = -\cos \beta_2, \quad \cos \gamma_1 = -\cos \gamma_2.$$

Conversely, if relations (9) are satisfied, the given lines are parallel and extend in opposite directions. Why?

If the two lines are *perpendicular*, it follows from formula (7) that,

$$(10)\qquad \cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2 = 0.$$

Conversely, if (10) is true, the lines will be perpendicular.

If l, m, n and l', m', n' are proportional to the direction cosines of two lines, the lines will be perpendicular if, and only if,

$$(11)\qquad ll' + mm' + nn' = 0.$$

They will be parallel if, and only if, the numbers l, m, n are proportional to l', m', n'. If any of the numbers l, m, n are zero, the corresponding numbers of the set l', m', n' must, of course, also be zero.

331. Point of Division. Let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be two given points and $P(x, y, z)$ any point on the segment P_1P_2 such that $\frac{P_1P}{PP_2} = \lambda$. If α, β, γ are the direction angles of this segment, it follows from § 326 that

$$P_1P \cos \alpha = x - x_1, \quad PP_2 \cos \alpha = x_2 - x.$$

Therefore,

$$\frac{P_1P \cos \alpha}{PP_2 \cos \alpha} = \frac{x - x_1}{x_2 - x} = \lambda,$$

or

$$(12)\qquad x = \frac{x_1 + \lambda x_2}{1 + \lambda}.$$

Similarly, we have

$$y = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z = \frac{z_1 + \lambda z_2}{1 + \lambda}. \tag{13}$$

It should be noticed that λ is positive if P lies within the segment P_1P_2, and negative if it lies without. By varying λ, the coördinates of any point ($\neq P_2$) on the line P_1P_2 may be obtained.

For the mid-point of P_1P_2 we have $\lambda = 1$ and, hence, the coördinates of the mid-point of P_1P_2 are

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}. \tag{14}$$

EXERCISES

1. Find the cosine of the angle between the two lines whose direction cosines are proportional to 2, 3, 1 and – 1, 4, 5.

2. Find the coördinates of the points of trisection of the segment $P_1(4, -1, 3)$, $P_2(-4, 7, 3)$.

3. Prove that the medians of the triangle whose vertices are (1, 2, 3), (3, 2, 1), (2, 1, 3) meet in a point.

4. Show that the following points are the vertices of a right triangle: (1, 0, 6), (7, 3, 4), (4, 5, – 2).

5. If two of the direction angles of a line are 45° and 60°, find the third direction angle.

6. Prove that the values $\alpha = 30°$, $\beta = 30°$ are impossible.

7. The direction cosines of a line are m, $2m$, $3m$. Find m.

8. Show that $(x-1)^2 + (y+2)^2 + (z-3)^2 = 9$ is the equation of a sphere whose center is at (1, – 2, 3) and whose radius is 3.

9. Express by an equation the fact that the point (x, y, z) is equidistant from (2, 1, 3) and (– 1, 4, 3).

10. Show that the points (3, 7, 2), (4, 3, 1), (1, 6, 3), (2, 2, 2) are the vertices of a parallelogram.

11. Prove by two methods that the points (3, 6, 4), (4, 13, 3), (2, – 1, 5) are collinear.

12. Show that the points (4, 3, – 4), (– 2, 3, 2), (– 2, 9, – 4) are the vertices of an equilateral triangle.

13. Find the coördinates of the point which divides the segment P_1P_2 in the ratio λ, given

(*a*) $P_1(2, 6, 8)$, $P_2(-1, 3, 5)$, $\lambda = 3$;
(*b*) $P_1(-2, -5, 8)$, $P_2(8, 0, -2)$, $\lambda = -2$;
(*c*) $P_1(3, -7, -9)$, $P_2(2, -2, -1)$, $\lambda = \frac{1}{3}$.

14. Prove that the medians of the triangle $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$ meet in the point

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

15. Prove that the lines joining the mid-points of the opposite edges of a tetrahedron pass through a common point and are bisected by that point.

16. Are the following points collinear: $(2, 1, 3)$, $(-2, -5, 3)$, $(1, 5, 7)$?

17. Find the direction cosines of the line that is equally inclined to the three axes.

18. Prove that the lines joining successively the middle points of the sides of any quadrilateral form a parallelogram.

19. Find the projection of the segment $P_1(1, 2, 3)$, $P_2(2, 1, 3)$ upon the line that passes through the points $P_3(-3, 5, -5)$, $P_4(8, -9, 12)$.

332. Locus of an Equation. We saw that in the plane the locus of the equation $f(x, y) = 0$ represents, in general, a curve. In an analogous way the equation $f(x, y, z) = 0$, in general, represents a *surface*. For, if we solve for z, we have $z = F(x, y)$ and from this equation, we see that we can find, corresponding to every point (x, y) in the xy-plane, one or more values of z (real or imaginary). The locus of the real points (x, y, z) is, in general, a surface, but may be a curve or a point. If there are no real values for x, y, z which satisfy the equation $f(x, y, z) = 0$, we say that the equation has no locus.

The locus of points satisfying the two conditions $f(x, y, z) = 0$ and $F(x, y, z) = 0$ is, in general, a *curve* in space, which is the intersection of the two surfaces represented by these equations.

333. The Plane. A plane is defined as a surface such that every point collinear with two points of the surface is itself a point of the surface.

We shall prove the following propositions:

(*a*) *Every equation of the first degree in x, y, and z represents a plane.*

(*b*) *Every plane is represented by an equation of the first degree in x, y, z.*

To prove (*a*), let $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ be any two points on the surface whose equation is $Ax + By + Cz + D = 0$. Then we have

$$Ax_1 + By_1 + Cz_1 + D = 0, \tag{15}$$

$$Ax_2 + By_2 + Cz_2 + D = 0. \tag{16}$$

Now let $P_3(x_3, y_3, z_3)$ be any point on the line P_1P_2. Then (if $P_3 \neq P_2$), there exists a value of $\lambda(\neq -1)$ such that

$$x_3 = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y_3 = \frac{y_1 + \lambda y_2}{1 + \lambda}, \quad z_3 = \frac{z_1 + \lambda z_2}{1 + \lambda}. \qquad (\S\ 331)$$

We wish to show that the coördinates of this point also satisfy the equation $Ax + By + Cz + D = 0$. By substitution in this equation we have

$$\frac{1}{1+\lambda}(Ax_1 + By_1 + Cz_1 + D) + \frac{\lambda}{1+\lambda}(Ax_2 + By_2 + Cz_2 + D) = 0. \tag{17}$$

Relation (17) is true, since it follows from (15) and (16) that each parenthesis vanishes separately. Therefore the surface defined by the equation $Ax + By + Cz + D = 0$ satisfies the definition of a plane.

To prove the statement (*b*), let π be any plane, and let OR be the perpendicular from O which meets π in P_1 (Fig. 262). The positive direction of OR will be taken from O to the plane. The direction angles of OR will be called α, β, γ

and the length OP_1 will be denoted by p.* Now if $P(x, y, z)$ is any point in the plane, we have, by § 326,

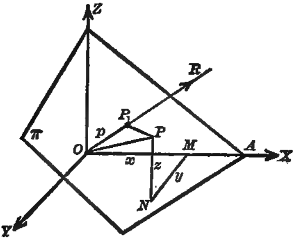

FIG. 262

(18) $$\text{Proj}_{OR}OP = \text{Proj}_{OR}OM + \text{Proj}_{OR}MN + \text{Proj}_{OR}NP.$$

Hence the equation

(19) $$x \cos \alpha + y \cos \beta + z \cos \gamma = p$$

is the equation of the plane. Why? It is seen to be an equation of the first degree in x, y, z. This form of the equation is called the ***normal form.***

It follows from the above that, if $Ax + By + Cz + D = 0$ is the equation of a plane, *the direction cosines of a line perpendicular to the plane are proportional to A, B, C.*

It is left as an exercise to prove that to reduce $Ax + By + Cz + D = 0$ to the normal form we must divide each term by $\pm\sqrt{A^2 + B^2 + C^2}$, the sign of the radical being chosen opposite to that of D if $D \neq 0$, the same as that of C if $D = 0$, the same as that of B if $C = D = 0$ or the same as that of A if $B = C = D = 0$.*

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* If the plane passes through the origin we shall suppose OR is directed upward, and hence $\cos\gamma > 0$ since $\gamma < \pi/2$. If the plane passes through the z-axis, then OR lies in the xy-plane and $\cos\gamma = 0$; in this case we shall suppose OR so directed that $\beta < \pi/2$ and hence $\cos\beta > 0$. Finally if the plane coincides with the yz-plane, the positive direction on OR shall be taken as that on OX.

334. The Angle Between Two Planes. The angle between two planes is defined to be the angle between two normals (*i.e.* perpendiculars) to the planes. Let $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ be the equations of the two planes. The direction cosines of their normals are then (§ 333),

$$\cos\alpha_1 = \frac{A_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}, \qquad \cos\alpha_2 = \frac{A_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\cos\beta_1 = \frac{B_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}, \qquad \cos\beta_2 = \frac{B_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}},$$

$$\cos\gamma_1 = \frac{C_1}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}}; \qquad \cos\gamma_2 = \frac{C_2}{\pm\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

If θ is the angle between these normals, then, from § 329,

$$(20) \qquad \cos\theta = \pm\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

If the planes are perpendicular, $\cos\theta = 0$, and we have

$$(21) \qquad A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

If the planes are parallel their normals are parallel. Hence, by § 330, their equations in normal form are

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p, \quad x\cos\alpha' + y\cos\beta' + z\cos\gamma' = p',$$

where either $\cos\alpha = \cos\alpha'$, $\cos\beta = \cos\beta'$, $\cos\gamma = \cos\gamma'$, or $\cos\alpha = -\cos\alpha'$, $\cos\beta = -\cos\beta'$, $\cos\gamma = -\cos\gamma'$. Therefore, if the two equations be written in the form

$$A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0,$$

the planes will be parallel if and only if

$$(22) \qquad A_2 = kA_1, \quad B_2 = kB_1, \quad C_2 = kC_1. \qquad (k \neq 0)$$

The equation of any plane parallel to $Ax + By + Cz + D = 0$ can therefore be written in the form $Ax + By + Cz + D' = 0$.

EXERCISES

1. Sketch the planes whose equations are (*a*) $x = 2$, (*b*) $y = 4$, (*c*) $z = -5$, (*d*) $2x + y = 1$, (*e*) $y - z = 0$.

2. How many arbitrary constants are there in the equation of the plane $Ax + By + Cz + D = 0$?

3. What is the general equation of a plane that passes through the origin?

4. What is the equation of the xy-plane? yz-plane? xz-plane?

5. What are the intercepts on the axes of the planes whose equations are
(*a*) $2x - 3y + z = 12$; (*b*) $x - y + z = 8$; (*c*) $x + y = 0$; (*d*) $5x - 7 = 0$?

6. Give three numbers proportional to the direction cosines of the normal to the plane $x + 2y - z = 9$. What are the direction cosines?

7. What is the normal equation of the plane $x - y + z = 9$?

8. What is the equation of the system of planes parallel to

$$2x - y + z = 1?$$

9. What is the equation of the plane that passes through the origin and is parallel to $2x - 3y + 7z = 5$?

10. Show that the planes $2x + 4y - z = 2$ and $4x - y + 4z = 7$ are perpendicular.

11. What is the equation of the plane parallel to $2x + 2y + z = 9$ and 5 units farther from the origin? 2 units nearer?

12. What is the distance between the parallel planes $2x + 2y + z = 9$ and $2x + 2y + z = 15$?

13. Find the equation of the plane passing through the points

(*a*) $(1, 2, 1)$, $(-1, 1, 0)$, $(0, 0, 1)$;
(*b*) $(2, 1, 3)$, $(1, 1, 2)$, $(-1, 1, 4)$;
(*c*) $(2, 2, 2)$, $(1, 1, -2)$, $(1, -1, 0)$;
(*d*) $(1, 1, -1)$, $(1, -1, 2)$, $(-2, -2, 2)$.

[HINT: Use the equation $Ax + By + Cz + D = 0$ and divide by any coefficient that is not zero.]

14. If $D \neq 0$ show that the equation $Ax + By + Cz + D = 0$ can be written in the form $x/a + y/b + z/c = 1$ where a, b, c are the intercepts made by the plane on the x, y, z axes respectively.

15. Show that the four points $(0, 0, 3)$, $(4, -3, -9)$, $(2, 1, 2)$, $(4, 3, 3)$ are coplanar, *i.e.* they lie in the same plane.

16. Find the equation of the plane that passes through the point P and is parallel to the plane α, when

(*a*) P is $(2, 1, 8)$ and α is $2x + 3y - 5z = 5$;
(*b*) P is $(1, 0, 0)$ and α is $2x + y + z = 1$;
(*c*) P is $(-2, -1, 6)$ and α is $3x - 5y - 2z = 3$.

17. Find the equation of the plane passing through the point P and perpendicular to the planes α and β when

(*a*) P is $(1, 1, 1)$, α is $2x - y - z = 4$, and β is $x + y + z = 1$;
(*b*) P is $(-1, 2, 1)$, α is $x + y - 3z = 3$, and β is $3x - 5y + 2z = 1$;
(*c*) P is $(0, 3, 4)$, α is $2x + 4y + z = 7$, and β is $2x - z + 3z = 2$.

18. Find the equation of the plane passing through the points P_1, P_2 and perpendicular to the plane α, when

(*a*) P_1 is $(1, 1, 1)$, P_2 is $(-1, 2, 1)$, and α is $2x - 3y - z = 2$;
(*b*) P_1 is $(0, 0, 1)$, P_2 is $(2, 1, 3)$, and α is $x + y - 6z = 0$;
(*c*) P_1 is $(2, 1, -3)$, P_2 is $(0, 4, 2)$, and α is $4x - y - z = 2$.

19. Prove that the distance from the plane $Ax + By + Cz + D = 0$ to the point (x_1, y_1, z_1) is $\pm \dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$.

20. Find the distance from the plane α to the point P when

(*a*) P is $(2, 1, 4)$ and α is $2x - 4y + z = 2$;
(*b*) P is $(2, 3, -1)$ and α is $2x + y + 25z - 2 = 0$;
(*c*) P is $(0, 0, 3)$ and α is $3x - 2y - 5z = 1$.

21. Prove that the equation of the plane which passes through the point (x_1, y_1, z_1) and is parallel to the plane $Ax + By + Cz + D = 0$ is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.

22. Prove that the equation of a plane which passes through the point (x_1, y_1, z_1) and is perpendicular to the plane $Ax + By + Cz + D = 0$ is $(Bz_1 - Cy_1)x + (Cx_1 - Az_1)y + (Ay_1 - Bx_1)z = 0$.

23. Find the cosines of the angles between the following pairs of planes

(*a*) $2x - 3y + z = 1$, $2x + z = 0$;
(*b*) $x - y - z = 2$, $y - 4z = 8$;
(*c*) $x + z = 3$, $4x + y + 3z = 5$.

24. Find the equation of the plane that passes through P_1, P_2 and makes an angle θ with the plane α, where

(*a*) P_1 is $(0, -1, 0)$, P_2 is $(0, 0, -1)$, α is $y + z - 7 = 0$, and θ is 120°;
(*b*) P_1 is $(1, 0, 1)$, P_2 is $(0, 1, 2)$, α is $x + 2y + 2z = 2$, and θ is 60°.

25. Find the equation of the locus of a point which moves so that its distance from the xy-plane is twice its distance from the x-axis.

26. Find the equation of the locus of a point whose distance from the plane $x + 2y - 5 = 0$ is twice its distance from the z-axis.

27. A point moves so that its distance from the origin is equal to its distance from the zx-plane. Find the equation of its locus.

335. Simultaneous Linear Equations. In § 70 we saw that three simultaneous linear equations in three unknowns have in general a single solution. We shall now show that three such simultaneous equations have either, (a) a single solution, or (b) an infinite number of solutions, or (c) no solution.

We shall prove this statement geometrically. Each equation represents a plane; the three planes may assume the following relative positions.

Case I. *No two of the planes are parallel or coincident.*

(a) The three planes may intersect in a single point; then there is a single solution of the three simultaneous equations.

(b) The three planes may intersect in a line; then there is an infinite number of solutions.

(c) The three planes may intersect so that the three lines of intersection are parallel; then there is no solution.

Case II. *Two of the planes are parallel but not coincident.*

In this case the three planes can have no point in common and the equations have no solution.

Case III. *Two of the planes are coincident.*

(a) The third plane may be parallel to the coincident planes, in which case there is no solution.

(b) The third plane may intersect the coincident planes, in which case there is an infinite number of solutions.

(c) The third plane may coincide with the coincident planes, in which case there is an infinite number of solutions.

336. Pencil of Planes. All the planes that pass through a given line are said to form a ***pencil of planes.*** If

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases} \tag{23}$$

are the equations of any two planes passing through the given line, then the equation of any *other* plane of the pencil can be written in the form

$$A_1x + B_1y + C_1z + D_1 + \lambda (A_2x + B_2y + C_2z + D_2) = 0, \tag{24}$$

where λ is a constant whose value determines the particular plane of the pencil. (See § 68.)

337. Bundle of Planes. All the planes that pass through a common point are said to form a ***bundle of planes,*** and this common point is called the ***center*** of the bundle. If

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \\ A_3x + B_3y + C_3z + D_3 = 0, \end{cases} \tag{25}$$

are the equations of any three planes passing through the center and not belonging to the same pencil, then the equation of any *other* plane of the bundle is

$$\begin{aligned} (A_1x+B_1y+C_1z+D_1)+\lambda_1(A_2x + B_2y + C_2z + D_2) \\ + \lambda_2(A_3x + B_3y + C_3z + D_3) = 0, \end{aligned} \tag{26}$$

where λ_1, λ_2 are constants whose values determine the position of the particular plane of the bundle. Why?

EXERCISES

1. Find the equation of the plane that passes through the intersection of the planes α and β and the point P, when

(*a*) α is $2x + 3y - z = 1$, β is $x + y - 2z = 2$, and P is $(1, 0, 2)$;
(*b*) α is $x + y + 2z = 0$, β is $4x - 2y - z = 1$, and P is $(2, 1, 1)$;
(*c*) α is $3x - 2y - z = 2$, β is $x - y + z = 3$, and P is $(1, 0, 1)$.

2. Show that the planes whose equations are $3x - 5y + 2 = 0$, $6x + y = 2z + 13$, $11y - 2z = 17$, belong to the same pencil.

3. What is the equation of the plane of the pencil whose axis is $2x - y + 5z + 2 = 0$, $4x - 3y + z = 1$, which is perpendicular to the plane $x = 0$? $y = 0$? $z = 0$?

4. Find the equation of the plane that passes through the intersection of the planes $2x + y - z + 1$, $3x - y - z = 2$ and is perpendicular to the plane $x + y - z = 1$.

5. Find the equation of the plane that passes through the point of intersection of the planes α, β, γ and the points P_1, P_2, when

(*a*) α is $2x + y = 1$, β is $x - z = 1$, γ is $2x - y + 2z = 3$, P_1 is $(1, 0, 1)$, and P_2 is $(2, 1, 1)$;

(*b*) α is $3x - y - z = 3$, β is $x - y + 2z = 1$, γ is $3x - 2y + z = 3$, P_1 is $(2, 1, 3)$, and P_2 is $(0, 8, 0)$.

338. Equations of a Straight Line. (*a*) The two simultaneous equations

$$\left\{\begin{array}{l} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{array}\right. \tag{27}$$

represent a line, the intersection of the two planes, provided the two planes are not parallel.

(*b*) A given point and a given direction determine a line. Let the given point be $P_1(x_1, y_1, z_1)$ and α, β, γ the given direction angles. If $P(x, y, z)$ is any other point on the line at a distance d from P_1, then by § 326, $d \cos\alpha = x - x_1$, $d \cos\beta = y - y_1$, $d \cos\gamma = z - z_1$. Hence we may write

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}, \tag{28}$$

which are the equations of the required straight line. These equations are known as the ***symmetric equations*** of a straight line. In these equations $\cos\alpha$, $\cos\beta$, $\cos\gamma$ can evidently be replaced by any three numbers proportional to them.

(c) Two distinct points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ determine a line. Any line through the point P_1 is of the form

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\cos\beta}=\frac{z-z_1}{\cos\gamma}.$$

Now the direction cosines of P_1P_2 are proportional to x_2-x_1, y_2-y_1, z_2-z_1. (§ 328.) Therefore the equations of the line through the points P_1, P_2 are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}. \tag{29}$$

We should note that in every case *two* equations are necessary to represent a line.

Example 1. Reduce to the symmetric form the equations of the straight line, $2x+y-z=3$, $x-y+2z=7$. Eliminating y between the two equations we have $3x+z=10$. Similarly, eliminating z we have $5x+y=13$. Solving these two equations for x and equating the values found, we have

$$\frac{x}{1}=\frac{y-13}{-5}=\frac{z-10}{-3}.$$

The line is seen to pass through the point (0, 13, 10) and to have direction cosines proportional to 1, − 5, − 3.

Example 2. Find the equations of the line that passes through the point (4, − 1, 3) and is perpendicular to the plane $2x-3y+4z=7$. The required line is parallel to any line perpendicular to the plane and hence its direction cosines are proportional to 2, − 3, 4 (§ 333). Therefore, the equation of the required line is

$$\frac{x-4}{2}=\frac{y+1}{-3}=\frac{z-3}{4}.$$

EXERCISES

1. Write the equations of the line that passes through the point P and whose direction cosines are proportional to a, b, c, where

(a) P is (1, 2, 1) and $a=2$, $b=-7$, $c=2$;
(b) P is (3, 0, −1) and $a=2$, $b=8$, $c=9$;
(c) P is (3, − 2, − 5) and $a=2$, $b=-9$, $c=3$.

2. Find the equations of the lines passing through the following pairs of points:

(a) $(2, 1, 4)$, $(12, 2, 8)$; (c) $(4, 3, 8)$, $(8, -2, 1)$;
(b) $(3, -5, -3)$, $(-3, 5, 7)$; (d) $(5, 2, 1)$, $(4, 7, -9)$.

3. Write in symmetric form the equations of the lines

(a) $2x - y + 3z = 8,\ 3x + 5y + z = 9$;
(b) $3x - y - z = 8,\ 4x + 6y - 3z = 3$;
(c) $5x + 8y + z = 3,\ 2x - y + z = 7$.

4. Find the equations of the line that passes through the point P and is perpendicular to the plane α, when

(a) P is $(2, 1, 7)$ and α is $3x - y + 4z = 9$;
(b) P is $(4, 2, -2)$, and α is $2x - 6y + 3z = 3$;
(c) P is $(-1, 6, 3)$, and α is $3x + 4y - z = 5$.

5. Find the equations of the line that passes through the point $(2, -1, 4)$ and is parallel to the line

$$\frac{x-3}{4} = \frac{y-7}{2} = \frac{z-7}{-3}.$$

6. Find in symmetric form the equations of the line that passes through the point $(2, -1, 4)$ and is parallel to the line $2x + y - z = 5$, $x - y + 3z = 4$.

7. Find the equation of the plane that passes through the point P and is perpendicular to the line l, when

(a) P is $(2, 5, 1)$, and l is $\frac{x-1}{3} = \frac{y+4}{6} = \frac{z+3}{5}$;

(b) P is $(-1, 4, 7)$, and l is $\frac{x+4}{5} = \frac{y-5}{-7} = \frac{z+3}{-2}$.

8. Find the equation of the plane that passes through $P(1, 5, 2)$ and is perpendicular to the line $3x - y + 2z = 8$, $x - y + 2z = 6$.

9 If θ is the angle between the two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$, and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, find $\cos\theta$.

10. Prove that the lines $\frac{x}{6} = \frac{y}{-2} = \frac{z}{-4}$, $\frac{x}{4} = \frac{y}{6} = \frac{z}{3}$ are perpendicular to each other.

CHAPTER XXII

QUADRATIC FUNCTIONS. QUADRIC SURFACES

339. The Sphere. If a point $P(x, y, z)$ moves so as to be always at a constant distance r $(r > 0)$ from a fixed point (h, k, l), the locus of P is called a ***sphere.*** The equation of this locus is

$$(1) \qquad (x-h)^2+(y-k)^2+(z-l)^2=r^2.$$

If this equation is expanded, it has the form

$$(2) \qquad x^2+y^2+z^2+Ax+By+Cz+D=0,$$

where A, B, C, D are constants depending upon the coördinates of the center and the length of the radius.

Conversely, an equation of the form (2), in general, represents a sphere, for it can be written in the form

$$(3) \qquad \left(x+\frac{A}{2}\right)^2+\left(y+\frac{B}{2}\right)^2+\left(z+\frac{C}{2}\right)^2=\frac{A^2}{4}+\frac{B^2}{4}+\frac{C^2}{4}-D,$$

which is a sphere if

$$\frac{A^2}{4}+\frac{B^2}{4}+\frac{C^2}{4}-D>0.$$

The center of the sphere is at the point $(-A/2, -B/2, -C/2)$, and the radius is

$$\sqrt{A^2/4+B^2/4+C^2/4-D}.$$

If the right-hand member of (3) is zero, the locus is the single point $(-A/2, -B/2, -C/2)$. If the right-hand member of (3) is less than zero, the equation has no locus. See § 206.

EXERCISES

1. Find the equation of the sphere whose center is at P and whose radius is r, when

(*a*) P is $(2, 1, 9)$, and $r = 6$;
(*b*) P is $(1, -8, 0)$, and $r = 2$;
(*c*) P is $(4, -9, -2)$, and $r = 7$.

2. Find the equations of the eight spheres tangent to the three coördinate planes and having a radius of 4.

3. Find the equation of the sphere which has the line joining $P(2, 6, 8)$ and $Q(4, 6, 6)$ as a diameter.

4. Discuss the locus of each of the following equations.

(*a*) $x^2 + y^2 + z^2 - 2x - 2y - 2z = 6$.
(*b*) $x^2 + y^2 + z^2 + 4x + 4y - 6z + 25 = 0$.
(*c*) $x^2 + y^2 + z^2 - 2x - 6y + 8z = 5$.
(*d*) $x^2 + y^2 + z^2 - 2z - 4y + 5 = 0$.

5. Find the locus of points the ratio of whose distances from $(0, 1, 0)$ and $(1, 2, 3)$ is 5.

6. Show that the equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = r^2$$

at the point (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 = r^2.$$

[HINT: The tangent plane is perpendicular to the radius.]

7. Find the equation of the sphere passing through the following four points.

(*a*) $(1, 2, 3)$, $(3, 1, 0)$, $(2, 1, 0)$, $(3, 4, 1)$.
(*b*) $(2, 1, 0)$, $(-1, -1, 0)$, $(3, 0, 2)$, $(0, 0, 0)$.

[HINT: Use the equation $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$ and determine the values of A, B, C, D.]

340. Cylinders. The surface generated by a straight line which moves parallel to a given line and always intersects a given fixed curve, is called a cylindrical surface or a ***cylinder***. The generating line in any of its positions is called an ***element*** of the cylinder.

Any algebraic equation in two cartesian coördinates represents in space a cylinder whose elements are parallel to the axis of the third variable.

For example, the equation

$$x^2 + y^2 = 4$$

represents in the xy-plane a circle (Fig. 263). But, the equation is satisfied by the coördinates of any point P which lies on a line parallel to the z-axis and passes through a point Q on the circle. Moreover, if QP moves parallel to the z-axis and continues to cut the circle the coördinates of P still satisfy the equation $x^2 + y^2 = 4$. The cylinder traced by the line QP is the locus of the equation $x^2 + y^2 = 4$.

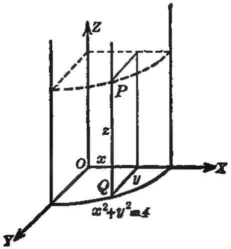

FIG. 263

It is clear that if a cylinder has its axis parallel to a coördinate axis, a section made by a plane perpendicular to that axis is a curve parallel and equal to the directing curve on the coördinate plane. Thus the section cut by the plane $z = 3$ from the hyperbolic cylinder whose equation is

$$x^2 - y^2 = 4,$$

is a hyperbola equal and parallel to the hyperbola in the xy-plane whose equation is $x^2 - y^2 = 4$.

341. The Projecting Cylinders of a Curve. A cylinder whose elements are parallel to one of the coördinate axes and always intersect a fixed curve in space, is called a *projecting cylinder* of the curve. The equations of the projecting cylinders may be found by eliminating in turn each of the variables x, y, z, from the equations of the curve. Why? The curve may often be constructed conveniently by means of two distinct projecting cylinders.

EXERCISES

1. Describe the locus of each of the following equations.

(*a*) $x = 2$.

(*b*) $2x^2 + y^2 = 8$.

(*c*) $3x - y = 9$.

(*d*) $y^2 = 4px$.

(*e*) $x^2 + z^2 = 9$.

(*f*) $x^2 = z$.

(*g*) $y^2 - z^2 = 1$.

(*h*) $yz = 5$.

(*i*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

(*j*) $3z^2 + 2x^2 = 6$.

(*k*) $y^2 = x^3$.

(*l*) $x - xz = 7$.

(*m*) $y^2 = x^2$.

2. Prove that $x^2 + 2xy + y^2 = 1 - z^2$ is the equation of a cylinder, the direction cosines of any element being proportional to (1, 1, 0).

3. Find the equations of the projecting cylinders of each of the following curves. Construct the curves as the intersection of two of these cylinders.

(*a*) $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 - z^2 = 0$.

(*b*) $x = 1$, $x^2 + y^2 + z^2 = 4$.

(*c*) $x^2 - y^2 = 4z$, $x^2 + y^2 = z$.

(*d*) $y^2 = x + z$, $z = x + y^2$.

(*e*) $z^2 = xy$, $x^2 = yz$.

342. Symmetry, Intercepts, Traces, Sections. If a given equation is unaffected by replacing x by $-x$ throughout, the locus is *symmetric* with respect to the yz-plane.

If a given equation is unaffected by replacing y by $-y$, the locus is *symmetric* with respect to the xz-plane.

If a given equation is unaffected by replacing z by $-z$, the locus is *symmetric* with respect to the xy-plane.

What would be a test for symmetry with respect to the x-axis? the y-axis? the z-axis? the origin?

The segments measured from the origin to where a surface cuts the axes are called the ***intercepts*** of the surface on the axes. To find the intercepts place two of the variables equal to zero and solve the resulting equation for the third variable. Why?

The sections of a surface made by the coördinate planes are called the ***traces*** of the surface (Fig. 264). To find the equations of the traces put each variable in turn in the given equation equal to zero. Why?

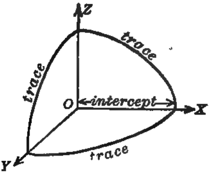

FIG. 264

The equations $f(x, y, z) = 0$ and $x = k$, a constant, are together the equations of the curve of intersection of the surface and a plane parallel to the yz-plane. Similarly sections parallel to the xy- and yz-planes may be found. If $k = 0$, the sections are the traces.

343. The Ellipsoid. The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{4}$$

is called an ***ellipsoid.*** It is symmetric with respect to the three coördinate planes, the three axes, and the origin. The intercepts on the x-, y-, z-axes are respectively $\pm a$, $\pm b$, $\pm c$ (Fig. 265).* The traces on the three coördinate planes are, respectively,

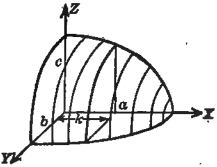

FIG. 265

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0;\quad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\ y=0;\quad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ x=0.$$

The sections of the ellipsoid by the plane $x = k$ is an ellipse whose equations are

$$\frac{y^2}{b^2\left(1-\frac{k^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{k^2}{a^2}\right)}=1,\quad x=k.$$

---

* The figure exhibits only that part of the surface lying in one octant, — that in which x, y, z are all positive.

The semi-axes of this ellipse are $b\sqrt{1-k^2/a^2}$, $c\sqrt{1-k^2/a^2}$. As $|k|$ increases from 0 to a, the axes of this elliptical section decrease. When $|k|=a$ the ellipse reduces to a point, and when $|k|>a$ the sections are imaginary. The surface lies therefore entirely between the planes $x=a$, $x=-a$. Similarly it may be shown that the surface is also bounded by the planes $y=b$, $y=-b$; $z=c$, $z=-c$.

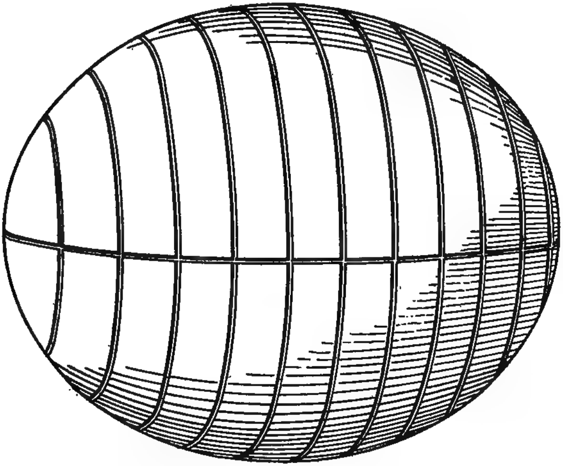

FIG. 266

A general idea of the appearance of an ellipsoid is given by Fig. 266, which represents a plaster model of this surface.

SPECIAL CASES. In general the semi-axes a, b, c are unequal, but it may happen that two or three of them are equal. If the three are equal, *i.e.* $a=b=c$, the surface is a *sphere.* If two are equal, for example, if $b=c$, the ellipsoid is called an ***ellipsoid of revolution***, for it can be generated by revolving the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$ about the x-axis.

344. Surfaces of Revolution. The surface generated by revolving a plane curve about a line in its plane is called a ***surface of revolution.*** The equation of the surface is readily found when the axis of revolution, *i.e.* the line about which the curve is revolved, is one of the coördinate axes.

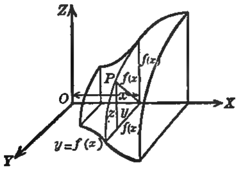

FIG. 267

Let $y = f(x)$ be the equation of the plane curve in the xy-plane and the x-axis the axis of revolution. As the curve $y=f(x)$ revolves about the x-axis, any point P on this curve describes a circle, whose center is on the x-axis and whose radius is equal to $f(x)$ (Fig. 267).

Therefore for any position of $P(x, y, z)$ we have,

$$y^2 + z^2 = [f(x)]^2$$

which is the equation of the required surface of revolution.

If the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = 0$ is revolved about the x-axis, the equation of the surface of revolution is

$$y^2 + z^2 = \frac{b^2}{a^2}[a^2 - x^2], \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1.$$

EXERCISES

1. Sketch and discuss each of the following ellipsoids.

(*a*) $9x^2 + 4y^2 + 16z^2 = 144$.

(*b*) $25x^2 + y^2 + z^2 = 100$.

(*c*) $x^2 + 8y^2 + 2z^2 = 16$.

2. Show that the ellipsoid in Ex. 1 (*b*) is an ellipsoid of revolution.

3. Find the equations of the ellipsoids formed by revolving the following ellipses about the axes mentioned.

(*a*) $9x^2 + 4y^2 = 36$, $z = 0$, x-axis.

(*b*) $9x^2 + 4y^2 = 36$, $z = 0$, y-axis.

(*c*) $9x^2 + z^2 = 9$, $y = 0$, z-axis.

(*d*) $25y^2 + 4z^2 = 100$, $x = 0$, y-axis.

4. When an ellipse is revolved about its major axis the ellipsoid generated is called a ***prolate spheroid***; when it is revolved about its minor axis, an ***oblate spheroid***. Which of the ellipsoids in Ex. 3 are oblate and which are prolate?

5. Describe the locus of each of the following equations.

$$(a)\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0. \qquad (b)\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1.$$

345. The Hyperboloid of One Sheet. The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{5}$$

is called a ***hyperboloid of one sheet***. It is symmetric with respect to each of the coördinate planes, each of the coördinate axes, and the origin. The intercepts on the x- and y-axes are $\pm a$ and $\pm b$ respectively, while the surface does not meet the z-axis (Fig. 268). The traces on the coördinate planes are, respectively,

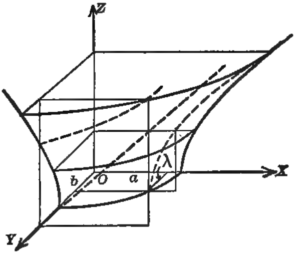

FIG. 268

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0;\ \frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ x=0;\ \frac{x^2}{a^2}-\frac{z^2}{c^2}=1,\ y=0.$$

Of these, the trace on the xy-plane is an ellipse, while the other two are hyperbolas.

The section of the surface made by the plane $z=k$, is an ellipse whose equations are

$$\frac{x^2}{a^2\left[1+\frac{k^2}{c^2}\right]}+\frac{y^2}{b^2\left[1+\frac{k^2}{c^2}\right]}=1,\ z=k.$$

This ellipse is real for all real values of k. The semi-axes are the smallest when $k = 0$ and increase without limit as $|k|$ increases.

The plane $y = \lambda$, $|\lambda| \neq b$, intersects the surface in the hyperbola

$$\frac{x^2}{a^2\left[1 - \frac{\lambda^2}{b^2}\right]} - \frac{z^2}{c^2\left[1 - \frac{\lambda^2}{b^2}\right]} = 1, \; y = \lambda.$$

If $|\lambda| < b$ the transverse axis is parallel to the x-axis, while if $|\lambda| > b$ it is parallel to the z-axis.

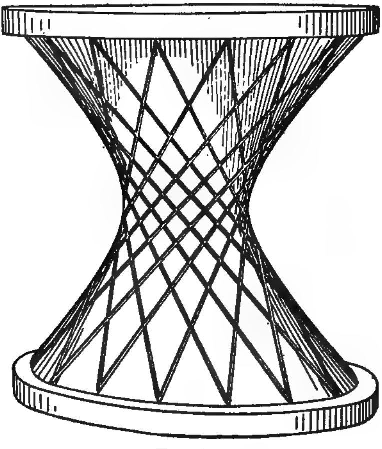

Fig. 269

A good idea of the appearance of this surface is given by Fig. 269, which represents a plaster model of a portion of the surface.

If $\lambda = b$, the section consists of the two straight lines

$$\frac{x}{a} + \frac{z}{c} = 0, \; y = b; \qquad \frac{x}{a} - \frac{z}{c} = 0, \; y = b.$$

If $\lambda = -b$, the section is the two lines

$$\frac{x}{a}+\frac{z}{c}=0,\ y=-b;\quad \frac{x}{a}-\frac{z}{c}=0,\ y=-b.$$

These four straight lines lie entirely upon the surface.

Similar considerations apply to the sections made by planes parallel to the yz-plane.

The form of one eighth of the surface is given in Fig. 268. The broken lines in that figure indicate three sections by the three planes

$$y=\lambda, \text{ for } |\lambda|<b,\ =b, \text{ and } >b.$$

Some of the straight lines on the surface are shown on the model represented by Fig. 269.

If $a=b$ the hyperboloid becomes a surface of revolution obtained by revolving the hyperbola $x^2/a^2 - z^2/c^2 = 1,\ y=0$ about its conjugate axis.

EXERCISES

1. Sketch and discuss each of the following surfaces.

(*a*) $x^2+4y^2-z^2=16$. (*b*) $9x^2+y^2-z^2=36$. (*c*) $4x^2+16y^2-z^2=64$.

2. Are any of the surfaces in Ex. 1 surfaces of revolution?

3. Show that

$$\frac{(x-2)^2}{9}+\frac{(y-1)^2}{4}-\frac{(z-3)^2}{1}=1$$

is the equation of a hyperboloid of one sheet whose center is at the point (2, 1, 3).

4. Show that $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are equations of hyperboloids of one sheet.

5. Find the equation of the hyperboloid of revolution formed by revolving each of the following hyperbolas about the axis specified.

(*a*) $9x^2-4y^2=36,\ z=0$, transverse axis.
(*b*) $9x^2-4y^2=36,\ z=0$, conjugate axis.
(*c*) $4y^2-z^2=16,\ x=0$, transverse axis.
(*d*) $4y^2-z^2=16,\ x=0$, conjugate axis.

346. Hyperboloid of Two Sheets. The surface represented by the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \tag{6}$$

is called a ***hyperboloid of two sheets.*** It is symmetric with respect to each of the coördinate planes, the coördinate axes and the origin. The intercepts on the x-axis are ± a, while the surface does not meet the y- or z-axis (Fig. 270).

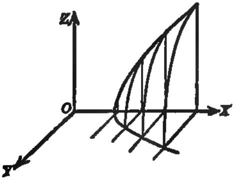

FIG. 270

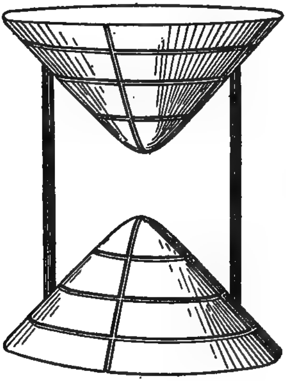

FIG. 271

The traces on the coördinate planes are, respectively,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ z=0; \qquad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1,\ y=0.$$

There is no trace on the yz-plane.

The plane $x = k$ intersects the surface in the curve whose equations, if $k \neq \pm a$, are

$$\frac{y^2}{b^2\left[\frac{k^2}{a^2}-1\right]}+\frac{z^2}{c^2\left[\frac{k^2}{a^2}-1\right]}=1,\ x=k.$$

If $|k| > a$ this curve is an ellipse; if $|k| = a$ it is a point. If $|k| < a$ the equations have no locus. All sections parallel to the xy- and xz-planes are hyperbolas.

A good idea of the appearance of this surface is given by Fig. 271, which represents a model of a portion of the surface.

If $b = c$ the hyperboloid becomes a surface of revolution formed by revolving the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad z=0,$$

about its transverse axis.

EXERCISES

1. Construct and discuss each of the following surfaces.

(*a*) $4x^2 - 9y^2 - 36z^2 = 144$.

(*b*) $x^2 - y^2 - z^2 = 1$.

(*c*) $9x^2 - 4y^2 - z^2 = 36$.

2. Are any of the surfaces in Ex. 1 surfaces of revolution?

3. Show that $-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ and $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are equations of hyperboloids of two sheets.

4. Find the equation of the hyperboloid of revolution formed by revolving each of the following hyperbolas about the axis specified.

(*a*) $\frac{x^2}{4}-\frac{y^2}{9}=1$, $z=0$, conjugate axis.

(*b*) $4y^2 - z^2 = 4$, $x = 0$, transverse axis.

(*c*) $2x^2 - 4z^2 = 1$, $y = 0$, conjugate axis.

347. The Elliptic Paraboloid. The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z \tag{7}$$

is called an ***elliptic paraboloid.*** It is symmetric with respect to the xz- and yz-planes, and the z-axis. The intercepts on all three axes are zero. The trace on the xy-plane is a point, namely the origin; the traces on the xz- and yz-planes are, respectively, the parabolas $x^2=a^2z$, $y=0$; $y^2=b^2z$, $x=0$ (Fig. 272).

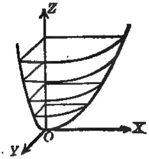

FIG. 272

Sections made by the planes $z=k$ $(k>0)$ are ellipses. Why? Those made by the planes $x=k$ and $y=k$, respectively, are parabolas. Why?

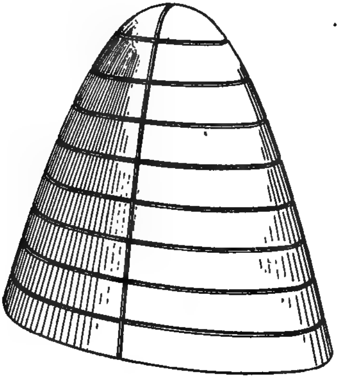

FIG. 273

Figure 273 represents a model of a portion of the surface. If $a=b$, the surface is a figure of revolution formed by revolving the parabola $x^2=a^2z$, $y=0$, about the z-axis.

348. The Hyperbolic Paraboloid. The surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z \tag{8}$$

is called a ***hyperbolic paraboloid.*** (See Fig. 274.) It is symmetric with respect to the xz- and yz-planes. All three intercepts are zero. The trace on the xy-plane is the pair of lines

$$\frac{x}{a} \pm \frac{y}{b} = 0, \ z = 0;$$

the traces on the xz- and yz-planes are, respectively, the parabolas

$$x^2 = a^2 z, \ y = 0; \qquad y^2 = -b^2 z, \ x = 0.$$

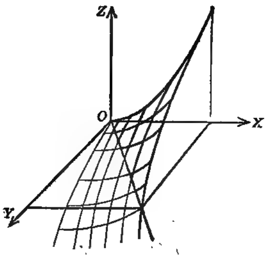

FIG. 274

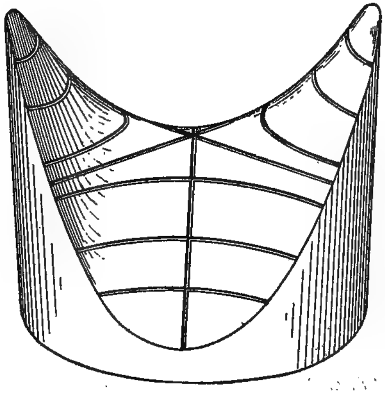

FIG. 275

Sections parallel to the xy-plane are hyperbolas, while those parallel to the xz- and yz-planes are parabolas. The form of the surface is shown in Fig. 275.

EXERCISES

1. Sketch and discuss each of the following surfaces.

(*a*) $x^2 + 4y^2 = 36z$.

(*b*) $2x^2 + z^2 = 16y$.

(*c*) $y^2 - z^2 = x$.

(*d*) $2x^2 - z^2 = -y$.

2. Sketch the surface $x^2 - 2x + y^2 - 4y = z - 5$.

349. The Cone. The surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \tag{9}$$

is called a ***cone***. It is symmetric with respect to the three coördinate planes, the three axes, and the origin. All three intercepts are zero. The trace on the xy-plane is a point, namely the origin. The traces on the xz- and yz-planes are respectively the pairs of lines $cx \pm az = 0$, $y = 0$; $cy \pm bz = 0$, $x = 0$ (Fig. 276). Sections parallel to the xy-plane are ellipses, while those parallel to the xz- and yz-planes are hyperbolas. If any point P (x_1, y_1, z_1) on the surface is connected with the origin, then the line OP lies entirely on the surface. For, $(\lambda x_1, \lambda x_2, \lambda x_3)$ are the coördinates of any point on this line (see § 331), and they are seen to satisfy the given equation (9), for all values of λ.

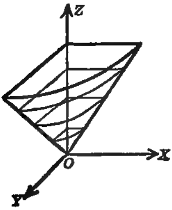

FIG. 276

If $a = b$ the cone is a cone of revolution.

EXERCISES

1. Construct and discuss each of the following surfaces.

(*a*) $x^2 + y^2 - z^2 = 0$. (*b*) $9x^2 + 4y^2 - 36z^2 = 0$. (*c*) $x^2 - y^2 + 4z^2 = 0$.

2. A point P moves so as to be equidistant from a plane and a line perpendicular to the plane. Find the equation of the locus of P.

3. A point P moves so that the sum of its distances from the three coördinate planes is equal to its distance from the origin. Find the equation of the locus of P.

350. Summary. The surfaces discussed are here enumerated for reference.

ELLIPSOID:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \text{(Figs. 265, 266, § 343)}$$

HYPERBOLOID OF ONE SHEET:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad \text{(Figs. 268, 269, § 345)}$$

HYPERBOLOID OF TWO SHEETS:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad \text{(Figs. 270, 271, § 346)}$$

ELLIPTIC PARABOLOID:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=z. \qquad \text{(Figs. 272, 273, § 347)}$$

HYPERBOLIC PARABOLOID:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z. \qquad \text{(Figs. 274, 275, § 348)}$$

QUADRIC CONE:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0. \qquad \text{(Fig. 276, § 349)}$$

QUADRIC CYLINDERS:

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,\quad y^2=4px. \qquad \text{(§ 340)}$$

It is beyond the scope of this book to prove that the general equation of the second degree in three variables x, y, z, can, in general, be reduced to one of the above types. Those interested in this problem will find it fully discussed in any standard textbook on solid analytic geometry.*

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* See, for example, SNYDER AND SISAM, *Analytic Geometry of Space*, Chapter 7.

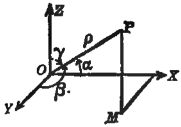

FIG. 277

351. Other Systems of Coördinates. Numerous systems of coördinates for determining the position of a point P in space have been devised. The most common of these systems are the rectangular, polar, spherical, and cylindrical. A brief account of the last three systems follows.

352. Polar Coördinates. Consider the line OP drawn from the origin O to any point P (Fig. 277). Let α, β, γ be the direction angles of OP, called the ***radius vector***, and let ρ be the length of the radius vector. The four quantities $\alpha, \beta, \gamma, \rho$ are called the ***polar coördinates*** of P.

Conversely, any four quantities $\alpha, \beta, \gamma, \rho$, with the restriction that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, determine a point whose polar coördinates are $\alpha, \beta, \gamma, \rho$.

Prove that the equations of transformation from rectangular to polar coördinates are,

$$x = \rho\cos\alpha,\quad y = \rho\cos\beta,\quad z = \rho\cos\gamma,\quad \rho^2 = x^2 + y^2 + z^2. \tag{10}$$

353. Spherical Coördinates. Any point P in space determines (Fig. 278) the radius vector $OP\,(= \rho)$, the angle ϕ between the radius vector and the z-axis, and the angle θ between the x-axis and the projection of the radius vector on the xy-plane. The quantities ρ, θ, ϕ are called the ***spherical coördinates*** of the point P. The angle ϕ is known as the ***colatitude***, and the angle θ as the ***longitude***.

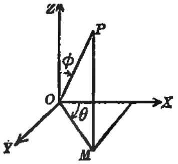

FIG. 278

Conversely, any three quantities ρ, θ, ϕ determine in space a point P whose spherical coördinates are ρ, θ, ϕ.

Prove that the equations of transformation from rectangular to spherical coördinates are,

$$x = \rho\sin\theta\cos\phi,\quad y = \rho\sin\theta\sin\phi,\quad z = \rho\cos\theta. \tag{11}$$

354. Cylindrical Coördinates. Any point P in space determines (Fig. 279) its distance z from the xy-plane and the polar coördinates r, θ of the point P' which is the projection of P on the xy-plane. These three quantities r, θ, z are called the ***cylindrical coördinates*** of P. Conversely, any three quantities r, θ, z determine a point whose cylindrical coördinates they are.

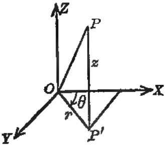

FIG. 279

Prove that the equations of transformations from rectangular to cylindrical coördinates are,

$$x = r\cos\theta,\quad y = r\sin\theta,\quad z = z. \tag{12}$$

EXERCISES

1. Express each of the following loci in spherical coördinates.

(*a*) $x^2 + y^2 + z^2 = 9$. (*b*) $x^2 + y^2 - 4z^2 = 0$. (*c*) $4x^2 + 9y^2 - z^2 = 36$.

2. Express each of the following loci in polar coördinates.

(*a*) $x^2 + y^2 + z^2 = 16$. (*b*) $x + y = 0$. (*c*) $2x^2 - y^2 - z^2 = 0$.

3. Express each of the following loci in cylindrical coördinates.

(*a*) $x^2 + y^2 = 9$. (*b*) $x^2 + y^2 + z^2 = 9$. (*c*) $z^2 - x^2 + y^2 = 6$.

4. Express the distance between two points in polar coördinates.

5. Find the polar, spherical, and cylindrical coördinates of the points whose rectangular coördinates are (2, 1, 4), (3, 3, 3).

6. What is the locus of points for which

(*a*) $\theta =$ a constant, $\phi =$ a constant (spherical coördinates)?
(*b*) $r =$ a constant, $\theta =$ a constant (cylindrical coördinates)?

7. Find the general equation of a plane in polar coördinates.

8. Find the general equation of a plane in spherical coördinates; in cylindrical coördinates.

9. Show that in polar coördinates a point may be regarded as the intersection of a sphere and three cones of revolution which have an element in common.

10. Show that in spherical coördinates a point may be regarded as the intersection of a sphere, a plane, and a cone of revolution which are mutually perpendicular.

11. The spherical coördinates of a point are 5, $\pi/4$, $\pi/6$; find its rectangular coördinates; its polar coördinates; its cylindrical coördinates.

TABLES

TO

FOUR DECIMAL PLACES

Squares and Cubes — Square Roots and Cube Roots

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	**51**	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	**52**	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	**53**	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	**54**	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	**55**	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	**56**	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	**57**	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	**58**	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	**59**	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	**61**	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	**62**	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	**63**	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	**64**	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	**65**	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	**66**	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	**67**	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	**68**	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	**69**	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	**71**	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	**72**	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	**73**	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	**74**	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	**76**	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	**77**	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	**78**	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	**79**	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	**81**	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	**82**	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	**83**	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	**84**	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	**85**	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	**86**	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	**87**	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	**88**	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	**89**	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	**90**	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	**91**	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	**92**	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	**93**	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	**94**	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	**95**	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	**96**	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	**97**	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	**98**	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	**99**	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.642

For a more complete table, see THE MACMILLAN TABLES, pp. 94–111.

CERTAIN CONVENIENT VALUES FOR $n = 1$ TO $n = 10$

n	$1/n$	$\sqrt{n}$	$\sqrt[3]{n}$	$n!$	$1/n!$	$\log_{10} n$
1	1.000000	1.00000	1.00000	1	1.0000000	0.000000000
2	0.500000	1.41421	1.25992	2	0.5000000	0.301029996
3	0.333333	1.73205	1.44225	6	0.1666667	0.477121255
4	0.250000	2.00000	1.58740	24	0.0416667	0.602059991
5	0.200000	2.23607	1.70998	120	0.0083333	0.698970004
6	0.166667	2.44949	1.81712	720	0.0013889	0.778151250
7	0.142857	2.64575	1.91293	5040	0.0001984	0.845098040
8	0.125000	2.82843	2.00000	40320	0.0000248	0.903089987
9	0.111111	3.00000	2.08008	362880	0.0000028	0.954242509
10	0.100000	3.16228	2.15443	3628800	0.0000003	1.000000000

LOGARITHMS OF IMPORTANT CONSTANTS

n = NUMBER	VALUE OF n	$\log_{10} n$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Napierian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812262
$\pi \div 180$ = radians in 1°	0.01745329	8.24187738
$\pi \div 10800$ = radians in 1′	0.0002908882	6.46372613
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811152	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37 (exact legal value)	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g (average value)	32.16 ft./sec./sec.	1.5073
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.907
cu. in. in 1 (U. S.) gallon	231 (exact legal value)	2.3636120
ft. lb. per sec. in 1 H. P.	550. (exact legal valne)	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
N	0	1	2	3	4	5	6	7	8	9	1	2	2	4	5	6	7	8	9

The proportional parts are stated in full for every tenth at the right-hand side. The logarithm of any number of four significant figures can be read directly by add-

N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ing the proportional part corresponding to the fourth figure to the tabular number corresponding to the first three figures. There may be an error of 1 in the last place.

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

RADIANS	DEGREES	SINE		TANGENT		COTANGENT		COSINE			
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$		
.0000	**0° 00′**	.0000	—	.0000	—	—	—	1.0000	.0000	**90° 00′**	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	**1° 00′**	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	**89° 00′**	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	**2° 00′**	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	**88° 00′**	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	**3° 00′**	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	**87° 00′**	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	**4° 00′**	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	**86° 00′**	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	**5° 00′**	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	**85° 00′**	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	**6° 00′**	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	**84° 00′**	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	**7° 00′**	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	**83° 00′**	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	**8° 00′**	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	**82° 00′**	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	**9° 00′**	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	**81° 00′**	1.4137
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	DEGREES	RADIANS
		COSINE		COTANGENT		TANGENT		SINE			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

RADIANS	DEGREES	SINE		TANGENT		COTANGENT		COSINE			
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$		
.1571	**9° 00′**	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	**81° 00′**	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	**10° 00′**	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	**80° 00′**	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	**11° 00′**	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	**79° 00′**	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	**12° 00′**	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	**78° 00′**	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	**13° 00′**	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	**77° 00′**	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	**14° 00′**	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	**76° 00′**	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	**15° 00′**	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	**75° 00′**	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	**16° 00′**	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	**74° 00′**	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	**17° 00′**	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	**73° 00′**	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	DEGREES	RADIANS
		COSINE		COTANGENT		TANGENT		SINE			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

RADIANS	DEGREES	SINE		TANGENT		COTANGENT		COSINE			
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$		
.3142	**18° 00′**	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	**72° 00′**	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	**19° 00′**	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	**71° 00′**	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	**20° 00′**	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	**70° 00′**	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	**21° 00′**	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	**69° 00′**	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	**22° 00′**	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	**68° 00′**	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	**23° 00′**	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	**67° 00′**	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	**24° 00′**	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	**66° 00′**	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	**25° 00′**	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	**65° 00′**	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	**26° 00′**	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	**64° 00′**	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	DEGREES	RADIANS
		COSINE		COTANGENT		TANGENT		SINE			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$		
.4712	**27° 00′**	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	**63° 00′**	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	**28° 00′**	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	**62° 00′**	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5167	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	**29° 00′**	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	**61° 00′**	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	**30° 00′**	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	**60° 00′**	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	**31° 00′**	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	**59° 00′**	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	**32° 00′**	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	**58° 00′**	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	**33° 00′**	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	**57° 00′**	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	**34° 00′**	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	**56° 00′**	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	**35° 00′**	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	**55° 00′**	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine		Tangent		Cotangent		Cosine			
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$		
.6283	36° 00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54° 00′	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	37° 00′	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	53° 00′	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	38° 00′	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	52° 00′	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	39° 00′	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	51° 00′	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	40° 00′	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	50° 00′	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	41° 00′	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	49° 00′	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	42° 00′	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	48° 00′	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	43° 00′	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	47° 00′	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	44° 00′	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	46° 00′	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	45° 00′	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	45° 00′	.7854
		Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Value	Log$_{10}$	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

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Analytic Geometry and Principles of Algebra

BY

ALEXANDER ZIWET

PROFESSOR OF MATHEMATICS, THE UNIVERSITY OF MICHIGAN

AND **LOUIS ALLEN HOPKINS**

INSTRUCTOR IN MATHEMATICS, THE UNIVERSITY OF MICHIGAN

Edited by EARLE RAYMOND HEDRICK

Cloth, viii + 369 pp., appendix, answers, index, 12mo, $1.75

This work combines with analytic geometry a number of topics traditionally treated in college algebra that depend upon or are closely associated with geometric sensation. Through this combination it becomes possible to show the student more directly the meaning and the usefulness of these subjects.

The idea of coördinates is so simple that it might (and perhaps should) be explained at the very beginning of the study of algebra and geometry. Real analytic geometry, however, begins only when the equation in two variables is interpreted as defining a locus. This idea must be introduced very gradually, as it is difficult for the beginner to grasp. The familiar loci, straight line and circle, are therefore treated at great length.

In the chapters on the conic sections only the most essential properties of these curves are given in the text; thus, poles and polars are discussed only in connection with the circle.

The treatment of solid analytic geometry follows the more usual lines. But, in view of the application to mechanics, the idea of the vector is given some prominence; and the representation of a function of two variables by contour lines as well as by a surface in space is explained and illustrated by practical examples.

The exercises have been selected with great care in order not only to furnish sufficient material for practice in algebraic work but also to stimulate independent thinking and to point out the applications of the theory to concrete problems. The number of exercises is sufficient to allow the instructor to make a choice.

To reduce the course presented in this book to about half its extent, the parts of the text in small type, the chapters on solid analytic geometry, and the more difficult problems throughout may be omitted.

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Elements of Analytic Geometry

BY

ALEXANDER ZIWET

PROFESSOR OF MATHEMATICS, THE UNIVERSITY OF MICHIGAN

AND **LOUIS ALLEN HOPKINS**

INSTRUCTOR IN MATHEMATICS, THE UNIVERSITY OF MICHIGAN

Edited by EARLE RAYMOND HEDRICK

As in most colleges the course in analytic geometry is preceded by a course in advanced algebra, it appeared desirable to publish separately those parts of the authors' "Analytic Geometry and Principles of Algebra" which deal with analytic geometry, omitting the sections on algebra. This is done in the present work.

In plane analytic geometry, the idea of function is introduced as early as possible; and curves of the form $y = f(x)$, where $f(x)$ is a simple polynomial, are discussed even before the conic sections are treated systematically. This makes it possible to introduce the idea of the derivative; but the sections dealing with the derivative may be omitted.

In the chapters on the conic sections only the most essential properties of these curves are given in the text; thus, poles and polars are discussed only in connection with the circle.

The treatment of solid analytic geometry follows more the usual lines. But, in view of the application to mechanics, the idea of the vector is given some prominence; and the representation of a function of two variables by contour lines as well as by a surface in space is explained and illustrated by practical examples.

The exercises have been selected with great care in order not only to furnish sufficient material for practice in algebraic work, but also to stimulate independent thinking and to point out the applications of the theory to concrete problems. The number of exercises is sufficient to allow the instructor to make a choice.

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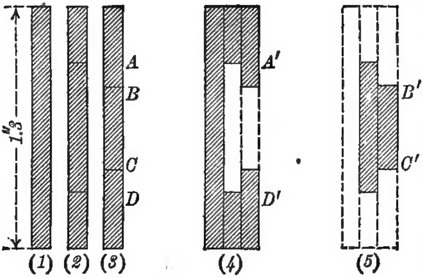
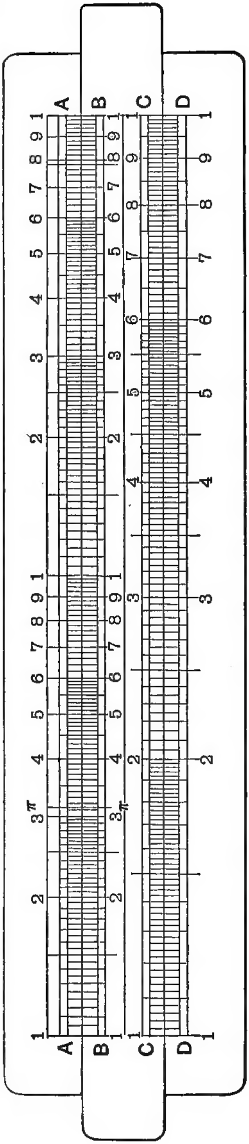


DIRECTIONS

A reasonably accurate slide-rule may be made by the student, for temporary practice, as follows. Take three strips of heavy stiff cardboard 1".3 wide by 6" long; these are shown in cross-section in (1), (2), (3) above. On (3) paste or glue the adjoining cut of the slide rule. Then cut strips (2) and (3) accurately along the lines marked. Paste or glue the pieces together as shown in (4) and (5). Then (5) forms the slide of the slide-rule, and it will fit in the groove in (4) if the work has been carefully done. Trim off the ends as shown in the large cut.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